Probability and Statistics

Exercise sheet 7

Exercise 7.1 (Getting the distribution of independent Binomials.)

We recall for this question the following (general) definition:

For X and Y two random variables, not necessarily defined on the same probability space, we say that X and Y have the same distribution (denoted $X \stackrel{d}{=} Y$) if

$$F_X = F_Y \text{ on } \mathbb{R} \tag{1}$$

with F_X and F_Y the cdf's of X and Y, respectively. Note that when X and Y are discrete, (1) is equivalent to $p_X = p_Y$, where p_X and p_Y are the pmf's of X and Y respectively.

Consider now X_1 and X_2 two *independent* random variables such that $X_1 \sim \text{Bin}(n_1, p)$ and $X_2 \sim \text{Bin}(n_2, p)$, with $n_1 \geq 1$, $n_2 \geq 1$ in $\mathbb{N}$ and $p \in (0, 1)$. We want to show that

$$X_1 + X_2 \sim \text{Bin}(n_1 + n_2, p).$$

(a) The hard way:

Assume without loss of generality that $n_1 \leq n_2$. Let $k \in \{0, ..., n_1 + n_2\}$. Show that

$$P(X_1 + X_2 = k) = \left[\sum_{j=\max(0,k-n_2)}^{\min(k,n_1)} \binom{n_1}{j} \binom{n_2}{k-j} \right] p^k (1-p)^{n_1+n_2-k}.$$

Using the fact that the pmf of any random variable with distribution $\text{Hypergeo}(n, D, N)$ has to add up to 1, show that

$$\sum_{j=\max(0,k-n_2)}^{\min(k,n_1)} \binom{n_1}{j} \binom{n_2}{k-j} = \binom{n_1+n_2}{k}$$

and conclude.

(b) A more elegant way:

On $\Omega = \{0, 1\}^{n_1+n_2}$ define the Bernoulli random variables $Y_1, ..., Y_{n_1}, Y_{n_i+1}, ..., Y_{n_1+n_2}$ such that they are all $\stackrel{\text{iid}}{\sim} \text{Bernoulli}(p)$.

Here, the Y_i have the natural definition that for each $i \in \{1, ..., n_1 + n_2\}$,

$$Y_i(\omega_1, ..., \omega_{n_1+n_2}) = \omega_i$$

for $(\omega_1, ..., \omega_{n_1+n_2}) \in \{0, 1\}^{n_1+n_2}$, and Ω is equipped with the probability measure P such that

$$P((\omega_1, ..., \omega_{n_1+n_2})) = p^{\omega_1}(1-p)^{1-\omega_1} ... p^{\omega_{n_1+n_2}}(1-p)^{1-\omega_{n_1+n_2}}$$

defined on $\mathcal{A} = 2^\Omega$.

Define the random variables

$$X_1' := Y_1 + ... + Y_{n_1},$$

$$X_2' := Y_{n_1+1} + ... + Y_{n_1+n_2}.$$

Show using a simple argument that

$$X_1 + X_2 \overset{d}{=} X_1' + X_2'$$

(without computing their cdf's or pmf's explicitly).

Conclude now that $X_1 + X_2 \sim \mathrm{Bin}(n_1 + n_2, p)$.

Solution 7.1

(a) The hard way:

Let $k \in \{0, ..., n_1 + n_2\}$.

$$\{X_1 + X_2 = k\} = \bigcup_{j=0}^{k} \{X_1 = j, X_2 = k - j\}$$

where the sets $\{X_1 = j, X_2 = k - j\}$ are pairwise disjoint for $j \in \{0, ..., k\}$.

Using independence of X_1 and X_2 we can write

$$P(X_1 + X_2 = k) = \sum_{j=0}^{k} P(X_1 = j)P(X_2 = k - j)$$

where

$$P(X_1 = j) = \begin{cases} \binom{n_1}{j} p^j (1-p)^{n_1 - j} & \text{if } 0 \leq j \leq n_1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P(X_2 = k - j) = \begin{cases} \binom{n_2}{k-j} p^{k-j} (1-p)^{n_2 - k + j} & \text{if } 0 \leq k - j \leq n_2, \\ 0 & \text{otherwise.} \end{cases}$$

The conditions

$$\begin{cases} 0 \leq j \leq n_1, \\ 0 \leq k - j \leq n_2, \end{cases}$$

are equivalent to

$$\begin{cases} 0 \leq j \leq n_1, \\ k - n_2 \leq j \leq k, \end{cases}$$

which are also equivalent to

$$\max(0, k - n_2) \leq j \leq \min(k, n_1).$$

Thus,

$$P(X_1 + X_2 = k) = \left[\sum_{j=\max(0,k-n_2)}^{\min(k,n_1)} \binom{n_1}{j} \binom{n_2}{k-j} \right] \times p^k (1-p)^{n_1+n_2-k}.$$

Consider now $Y \sim \text{Hypergeo}(k, n_1, n_1 + n_2)$. We know that the pmf of Y is given by

$$p(y) = P(Y = y) = \frac{\binom{n_1}{y}\binom{n_1+n_2-n_1}{k-y}}{\binom{n_1+n_2}{k}}$$

for $\max(0, k - n_2) \le y \le \min(k, n_1)$. Thus,

$$\sum_{y=\max(0,k-n_2)}^{\min(k,n_1)} p(y) = 1,$$

which gives the identity

$$\sum_{j=\max(0,k-n_2)}^{\min(k,n_1)} \binom{n_1}{j}\binom{n_2}{k-j} = \binom{n_1+n_2}{k}.$$

It follows that

$$P(X_1 + X_2 = k) = \binom{n_1+n_2}{k} p^k (1-p)^{n_1+n_2-k}$$

for $k \in \{0, ..., n_1 + n_2\}$, from which we conclude that

$$X_1 + X_2 \sim \text{Bin}(n_1 + n_2, p).$$

(b) A more elegant way:

If we compute the probability mass functions of $X_1 + X_2$ and $X_1' + X_2'$, we clearly see that they must be equal. This follows by the definition of X_1, X_2 and X_1', X_2' and the fact that $X_1 \perp\!\!\!\perp X_2$ and $X_1' \perp\!\!\!\perp X_2'$.

Then, $X_1 + X_2$ has the same distribution as the sum of $n_1 + n_2$ i.i.d Bernoulli(p), and therefore must have a $\text{Bin}(n_1 + n_2, p)$ distribution.

Exercise 7.2

(a) Let

$$f(x) := \frac{1}{x^k} \mathbb{1}_{x \in [1,+\infty)}.$$

For what value of k, if any, is f a density function?

(b) Give an example of a density function f such that $c\sqrt{f}$ cannot be a density function for any $c > 0$.

(c) Let

$$f(x) = c\,|x|\,(1 - x^2)\mathbb{1}_{|x|\le 1}.$$

1. Find $c > 0$ so that f is a density function.
2. Find the cdf corresponding to this density.
3. Compute $P(X < -\frac{1}{2})$ and $P(|X| \le \frac{1}{2})$.

Solution 7.2

(a) From the lectures, we know that for a measurable $f \geq 0$ to be a density on $\mathbb{R}$, it has to satisfy

$$\int_{\mathbb{R}} f(t)dt = 1.$$

f is measurable since it is piecewise continuous. We see that for f to be at all integrable, k has to be strictly larger than 1.

Let $k > 1$. Then

$$\int_{\mathbb{R}} \frac{dx}{x^k} \mathbb{1}_{x \in [1,\infty)} = \int_1^{\infty} \frac{dx}{x^k} = \frac{1}{k-1}.$$

Then $k - 1 = 1 \Rightarrow k = 2$. Hence, $k = 2$ is the only possibility for f to be a density.

(b) If we take

$$f(x) = \frac{1}{x^2} \mathbb{1}_{x \in [1,+\infty)}$$

then

$$\sqrt{f(x)} = \frac{1}{x} \mathbb{1}_{x \in [1,+\infty)}$$

which is not integrable, which implies that there is no $c \in \mathbb{R}$ such that $c\sqrt{f}$ is a density.

(c)

$$f(x) = c|x|(1 - x^2)\mathbb{1}_{|x| \leq 1}.$$

1.

$$\begin{aligned} 1 &= \int_{\mathbb{R}} f(x)dx \\ &= 2c \int_0^1 x(1 - x^2)dx \\ &= 2c \left[-\frac{(1 - x^2)^2}{4} \right]_0^1 \\ &= \frac{c}{2}(0 + 1) \\ &= \frac{c}{2}. \end{aligned}$$

Thus, $c = 2$.

2. By definition, the cdf of the random variable whose density is f is given by

$$\begin{aligned} F(x) = P(X \leq x) &= \int_{-\infty}^{x} f(t)dt \\ &= \begin{cases} 0 & \text{if } x < -1, \\ 2\int_{-1}^{x} |t|(1 - t^2)dt & \text{if } -1 \leq x < 1, \\ 1 & \text{if } x \geq 1. \end{cases} \end{aligned}$$

In the interval $[-1, 1]$, there are two further cases:

$$F(x) = \begin{cases} 2\int_{-1}^{x}(-t)(1-t^2)dt & \text{if } -1 \le x \le 0, \\ 2\int_{-1}^{0}(-t)(1-t^2)dt + 2\int_{0}^{x} t(1-t^2)dt & \text{if } 0 \le x \le 1, \end{cases}$$

where

$$2\int_{-1}^{x}(-t)(1-t^2)dt = \left[\frac{(1-t^2)^2}{2}\right]_{-1}^{x} = \frac{(1-x^2)^2}{2}$$

and

$$2\int_{0}^{x} t(1-t^2)dt = \left[-\frac{(1-t^2)^2}{2}\right]_{0}^{x} = \frac{1}{2}(1-(1-x^2)^2).$$

Thus

$$F(x) = \begin{cases} \frac{(1-x^2)^2}{2} & \text{if } -1 \le x \le 0, \\ 1 - \frac{(1-x^2)^2}{2} & \text{if } 0 \le x \le 1. \end{cases}$$

To conclude,

$$F(x) = \begin{cases} 0 & \text{if } x < -1, \\ \frac{(1-x^2)^2}{2} & \text{if } -1 \le x < 0, \\ 1 - \frac{(1-x^2)^2}{2} & \text{if } 0 \le x < 1, \\ 1 & \text{if } x \ge 1. \end{cases}$$

As a quick check of monotonicity, one can observe that $(1-x^2)^2$ decreases as $|x|$ increases, and therefore each of the branches is monotonically increasing. Moreover, one sees that $F(0) = \frac{1}{2}$ is the same on the two middle branches.

3.

$$\begin{aligned} P\left(X < -\frac{1}{2}\right) &= P\left(X \le -\frac{1}{2}\right) \\ &= F\left(-\frac{1}{2}\right) \\ &= \left(1 - \frac{1}{4}\right)^2 \times \frac{1}{2} \\ &= \frac{9}{32} \end{aligned}$$

(in the first step we used the fact that X is absolutely continuous, implying that $P(X = a) = 0$ for any $a \in \mathbb{R}$).

$$\begin{aligned} P\left(|X| \le \frac{1}{2}\right) &= F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) \\ &= 1 - F\left(-\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) \\ &= 1 - 2 \times \frac{9}{32} \\ &= 1 - \frac{9}{16} = \frac{7}{16}. \end{aligned}$$

Exercise 7.3 (On moment generating functions).

For a random variable X, the moment generating function is defined as

$$\Psi_X(t) := E[e^{tX}]$$

for any $t \in \mathbb{R}$ for which this expectation is finite.

In this question, we will assume that for two random variables X and Y (*not* necessarily defined on the same probability space), we have the equivalence

$$X \stackrel{d}{=} Y \Leftrightarrow \Psi_X = \Psi_Y \text{ on } (a, b)$$

for some non-empty open interval (a, b) containing 0. (One can prove this equivalence indeed holds, as long as such an interval (a, b) exists where the moment generating functions are defined).

(a) Let $X \sim \text{Bin}(n, p)$. Compute Ψ_X (on its domain of definition). If $X_1 \sim \text{Bin}(n_1, p)$ and $X_2 \sim \text{Bin}(n_2, p)$ are independent, compute $\Psi_{X_1+X_2}$. Can you conclude again that $X_1 + X_2 \sim \text{Bin}(n_1 + n_2, p)$?

(b) Let $X \sim \mathcal{N}(0, 1)$. Compute Ψ_X (on its domain of definition). If $X_1, X_2, ..., X_n$ are $\stackrel{\text{iid}}{\sim} \mathcal{N}(0, 1)$, compute $\Psi_{X_1+...+X_n}$. What is then the distribution of $X_1 + ... + X_n$?

(c) Let $X \sim \text{Exp}(\lambda)$ for some $\lambda > 0$. Recall that this means that the density of X is given by

$$f(x) = \lambda e^{-\lambda x} \mathbb{1}_{x \in (0,\infty)}.$$

Compute Ψ_X (on its domain of definition).

(d) Let $X \sim G(\alpha, \beta)$, $\alpha > 0$, $\beta > 0$. Compute Ψ_X (on its domain of definition). If $X_1, ..., X_n$ are independent such that $X_i \sim G(\alpha_i, \beta)$ for $\alpha_i > 0, \beta > 0$, compute $\Psi_{X_1+...+X_n}$. What is the distribution of $X_1 + ... + X_n$?

Solution 7.3

(a)

$$X \sim \text{Bin}(n, p).$$

We can calculate the moment generating function as follows:

$$\begin{aligned}
\Psi_X(t) &= E\left(e^{tX}\right) \\
&= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(e^t p\right)^k (1-p)^{n-k} \\
&= \left(e^t p + 1 - p\right)^n \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Let $X_1 \sim \text{Bin}(n_1, p)$ and $X_2 \sim \text{Bin}(n_2, p)$, such that $X_1 \perp\!\!\!\perp X_2$. Then,

$$\begin{aligned}
\Psi_{X_1+X_2}(t) &= E\left[e^{t(X_1+X_2)}\right] \\
&= E\left[e^{tX_1} e^{tX_2}\right] \\
&= E\left(e^{tX_1}\right) E\left(e^{tX_1}\right) \quad \text{(by independence)} \\
&= \left(e^t p + 1 - p\right)^{n_1} \left(e^t p + 1 - p\right)^{n_2} \\
&= \left(e^t p + 1 - p\right)^{n_1+n_2} \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Comparing with our earlier formula, and since the moment generating function determines the distribution (by our assumption), we conclude that $X_1 + X_2 \sim \text{Bin}(n_1 + n_2, p)$.

(b) $X \sim \mathcal{N}(0,1)$. We calculate

$$\begin{aligned}
\Psi_x(t) &= E\left[e^{tX}\right] \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{tx} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2tx+t^2)} dx \times e^{\frac{t^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} dx \times e^{\frac{t^2}{2}} \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \times e^{\frac{t^2}{2}} \quad (\text{with } z = x - t) \\
&= e^{\frac{t^2}{2}}
\end{aligned}$$

(in the last step we notice that we are just integrating the density of a $\mathcal{N}(0,1)$ distribution).

If $X_1, ..., X_n$ are $\overset{\text{iid}}{\sim} \mathcal{N}(0,1)$ we have

$$\begin{aligned}
\Psi_{X_1+...X_n}(t) &= E\left[e^{tX_1}...e^{tX_n}\right] \\
&= \prod_{j=1}^{n} E\left[e^{tX_j}\right] \quad \text{by independence} \\
&= \left(e^{\frac{t^2}{2}}\right)^n = e^{\frac{nt^2}{2}}.
\end{aligned}$$

To answer the question about the distribution of $X_1 + ... + X_n$, note that if we go back to the situation with one random variable $X \sim \mathcal{N}(0,1)$, we have for any $\sigma > 0$

$$\begin{aligned}
\Psi_{\sigma X} &= E\left[e^{t\sigma X}\right] \\
&= \Psi_X(t\sigma) \\
&= e^{\frac{t^2\sigma^2}{2}} \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Hence the distribution of $X_1 + ... + X_n$ is equal to the distribution of $\sqrt{n}Z$ with $Z \sim \mathcal{N}(0,1)$. We will see in the lectures that this means that $X_1 + ...X_n \sim \mathcal{N}(0,n)$.

(c) $X \sim \text{Exp}(\lambda)$ for some $\lambda \in (0, +\infty)$. We have

$$\begin{aligned}
\Psi_X(t) &= \int_0^{\infty} \lambda e^{-\lambda x} e^{tx} dx \\
&= \lambda \int_0^{\infty} e^{(t-\lambda)x} dx \\
&< \infty \quad \text{if and only if } t - \lambda < 0
\end{aligned}$$

in which case,

$$\int_0^\infty e^{(t-\lambda)x}dx = \frac{1}{\lambda - t}.$$

Thus,

$$\Psi_X(t) = \frac{\lambda}{\lambda - t}$$

for all $t \in (-\infty, \lambda)$.

(d) $X \sim G(\alpha, \beta)$, $\alpha > 0, \beta > 0$.
We have

$$\begin{aligned}\Psi_X(t) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{xt} x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{(t-\beta)x} dx \\ &< \infty \quad \text{if and only if } t < \beta\end{aligned}$$

in which case,

$$\begin{aligned}\int_0^\infty x^{\alpha-1} e^{(t-\beta)x} dx &= \frac{(\beta-t)^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x} dx \times \frac{\Gamma(\alpha)}{(\beta-t)^\alpha} \\ &= \frac{\Gamma(\alpha)}{(\beta-t)^\alpha}\end{aligned}$$

using the fact that

$$\frac{\gamma^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\gamma c} \mathbb{1}_{x \in (0,\infty)}$$

is a density (of $G(\alpha, \gamma)$) provided that $\alpha > 0$ and $\gamma > 0$.
Thus,

$$\Psi_X(t) = \frac{\beta^\alpha}{(\beta-t)^\alpha} = \left(\frac{\beta}{\beta - t}\right)^\alpha \quad \forall t \in (-\infty, \beta).$$

Let $X_1, ..., X_n$ be independent random variables with each $X_i \sim G(\alpha_i, \beta)$ for $i \in \{1, ..., n\}$.
We have

$$\begin{aligned}\Psi_{X_1+\ldots X_n}(t) &= \prod_{i=1}^n E[e^{tX_i}] \\ &= \prod_{i=1}^n \Psi_{X_i}(t) \quad \forall t \in (-\infty, \beta) \\ &= \prod_{i=1}^n \left(\frac{\beta}{\beta-t}\right)^{\alpha_i} \quad \forall t \in (-\infty, \beta) \\ &= \left(\frac{\beta}{\beta-t}\right)^{\sum_{i=1}^n \alpha_i} \quad \forall t \in (-\infty, \beta)\end{aligned}$$

from which we conclude that $X_1 + ...X_n \sim G(\sum_{i=1}^n \alpha_i, \beta)$.

Exercise 7.4 (Optional).
The first goal of this exercise is to show that:

(a) $X : (\Omega, \mathcal{A}, P) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is an absolutely continuous random variable with density f if and only if

$$\forall x \in \mathbb{R} \quad F(x) = \int_{(-\infty,x]} f d\lambda = \int_{-\infty}^{x} f(t)dt.$$

1. Show that the condition is necessary.
2. To show that it is sufficient, define the probability measure

$$\mu_1(B) = \int_B f d\lambda = \int_B f(t)dt, \ \forall B \in \mathcal{B}_{\mathbb{R}}.$$

Use Carathéodory's extension theorem to show that μ_1 and P_X have to be equal and conclude.

The second goal is to show the following:

(b) A measurable function $f \geq 0$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is the density of some absolutely continuous random variable if and only if

$$\int_{\mathbb{R}} f(t)dt = 1.$$

1. Show that this condition is necessary.
2. To show that it is sufficient, define

$$F(x) := \int_{-\infty}^{x} f(t)dt, \ x \in \mathbb{R}.$$

Show that:

- F is non-decreasing on $\mathbb{R}$,
- $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$,
- F is right-continuous on $\mathbb{R}$.

Conclude from this that F has to be the cdf of some random variable X (simply invoke a result from the lecture; no need to give a formal proof).

Finally, show that this X has to be absolutely continuous with density f (for this use (a)).

Solution 7.4

(a) 1. That the condition is necessary is rather obvious since for $B \in (-\infty, x]$

$$P_X(B) = F(x) = \int_B f d\lambda = \int_{-\infty}^{x} f(t)dt.$$

2. Define the probability measure

$$\mu_1(B) = \int_B f d\lambda = \int_B f(t)dt.$$

Take now $B = \bigcup_{i\in\mathbb{N}}(a_i, b_i]$ for some $a_1, a_2, ..., b_1, b_2, ... \in \mathbb{R}$ such that $a_i < b_i$ for each $i \in \mathbb{N}$ and the $(a_i, b_i]$ are all pairwise disjoint.
Then, .

$$\begin{aligned}
\mu_1(B) &= \sum_{i\in\mathbb{N}} \mu_1((a_i, b_i]) \quad \text{by } \sigma \text{ -additivity of } \mu_1 \\
&= \sum_{i\in\mathbb{N}} \int_{(a_i,b_i]} f(t)dt \quad \text{by definition of } \mu_1 \\
&= \sum_{i\in\mathbb{N}} (F(b_i) - F(a_i)) \\
&= \sum_{i\in\mathbb{N}} P(X \in (a_i, b_i]) \\
&= P\left(X \in \bigcup_{i\in\mathbb{N}} (a_i, b_i]\right) \quad \text{by } \sigma\text{-additivity of } P \\
&= P_X(B).
\end{aligned}$$

Hence, μ_1 and P_X have to be equal on the ring

$$\mathcal{R} = \left\{ \bigcup_{i\in\mathbb{N}} (a_i, b_i] : -\infty < a_i < b_i < \infty \right\}.$$

Now, $\mathcal{B}_\mathbb{R} = \sigma(\mathcal{R})$ and μ_1 is σ-finite (since it is finite as a probability measure). Using Carathéodory's extension theorem (see Theorem 3.3 in the Measure and Integration script, p.20) we know that there exists a unique measure μ on $\mathcal{B}_\mathbb{R}$ such that $\mu = \mu_1$ on $\mathcal{R}$.
We conclude that $\mu_1 = P_X$ and that X is absolutely continuous with density f.

(b) 1. This is easy to see, since by definition

$$\begin{aligned}
1 &= P(\Omega) \\
&= P(X \in \mathbb{R}) \\
&= \int_{-\infty}^{\infty} f(t)dt.
\end{aligned}$$

2.

- Let $x \le y$. We have $\mathbb{1}_{(-\infty,x]} \le \mathbb{1}_{(-\infty,y]}$ which implies that

$$\int_{(-\infty,x]} f(t)dt \le \int_{(-\infty,y]} f(t)dt,$$

that is, F is non-decreasing.

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$$F(x) = \int_\mathbb{R} f(t)\mathbb{1}_{(-\infty,x]}(t)dt$$

with $0 \leq f \mathbb{1}_{(-\infty,x]} \leq f$.
Since f is integrable and

$$\lim_{x \to +\infty} f(t) \mathbb{1}_{(-\infty,x]}(t) = 0$$

for any $t \in \mathbb{R}$, it follows from the dominated convergence theorem that

$$\lim_{x \to -\infty} F(x) = \int_{\mathbb{R}} \lim_{x \to -\infty} f(t) \mathbb{1}_{(-\infty,x]}(t) dt = 0.$$

Similarly, we can show that

$$\begin{aligned} \lim_{x \to +\infty} F(x) &= \int_{\mathbb{R}} \lim_{x \to +\infty} f(t) \mathbb{1}_{(-\infty,x]}(t) dt \\ &= \int_{\mathbb{R}} f(t) dt = 1. \end{aligned}$$

- That F is right-continuous follows from the fact that it is continuous, which has been proved in lectures.

We conclude that there exists a random variable X (on some probability space $(\Omega, \mathcal{A}, P)$) with cdf equal to this F.

To show that this existing random variable is absolutely continuous with density f, it suffices to use (a) to conclude that we must have that

$$P_X(B) = P(X \in B) = \int_B f(t) dt$$

for any $B \in \mathcal{B}$.