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Algebraic Topology II

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Problem set 1

1. Consider the space $X := \Delta^n/\partial\Delta^n \approx S^n$ for $n \geq 1$. The quotient map $\sigma_n : \Delta^n \to X$, viewed as a singular n-simplex, is a cycle in $S_n(X, *)$. Show that $[\sigma_n]$ generates $H_n(X, *; \mathbb{Z}) \cong \widetilde{H}_n(X; \mathbb{Z})$.

2. Consider the space $Y \approx S^n$ obtained by gluing two copies $\Delta^n_\pm$ of Δ^n along their boundaries (using the identity map). Consider the obvious singular simplices $\tau_\pm : \Delta^n \to Y$ mapping to the subsets $\Delta^n_\pm \subset Y$. Check that $\tau_+ - \tau_-$ is a cycle and prove that $[\tau_+ - \tau_-]$ generates $\widetilde{H}_n(Y; \mathbb{Z})$.

   Hint: Use the Mayer-Vietoris sequence.

3. Suppose you know that $H_k(\mathbb{R}P^n; \mathbb{Z}_2) = 0$ for all $k > n$. In the lecture you have seen a long exact sequence (also known as the *Smith-sequence*) for the 2:1-covering $S^n \to \mathbb{R}P^n$. Use the Smith-sequence for this covering to compute $H_k(\mathbb{R}P^n; \mathbb{Z}_2)$ for $0 \leq k \leq n$.

4. Let $f : \mathbb{R}P^n \to \mathbb{R}P^m$ be any map, where $n > m > 0$. Show that the induced map $f_\# : \pi_1(\mathbb{R}P^n) \to \pi_1(\mathbb{R}P^m)$ is trivial.

5. Show that $\mathbb{R}P^2$ is not a retract of $\mathbb{R}P^3$.

6. The Borsuk-Ulam theorem says that for every map $f : S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ such that $f(x) = f(-x)$. Give a proof of the theorem based on the following steps:

   (a) Let $g : S^n \to S^n$ be an odd map, i.e., such that $g(-x) = -g(x)$ for all $x \in S^n$. Show that g induces a natural homomorphism from the Smith sequence for $S^n \to \mathbb{R}P^n$ to itself in which all squares commute.

   (b) Conclude that every odd $g : S^n \to S^n$ has odd degree.

   (c) Conclude the proof of the theorem.

7. Use Borsuk-Ulam to prove that whenever there exists a map $\phi : S^n \to S^m$ which is equivariant with respect to the antipodal maps, then $n \leq m$.

8. Use Borsuk-Ulam to prove the following: Given Lebesgue measurable bounded subsets $A_1, \ldots, A_m$ of $\mathbb{R}^m$, there exists a hyperplane $H \subset \mathbb{R}^m$ which divides each A_i into pieces of equal measure. (This is known as the "Ham Sandwich Theorem".)