Problem set 2

1. Let R be a commutative ring and let M be an R-module. Show that for every exact sequence of R-modules $U \xrightarrow{f} V \xrightarrow{g} W \to 0$ the sequence
$$M \otimes U \xrightarrow{\mathrm{id}\otimes f} M \otimes V \xrightarrow{\mathrm{id}\otimes g} M \otimes W \to 0$$
is exact. *Hint:* To prove exactness at $M \otimes V$, construct a left-inverse for an appropriate map $M \otimes V/\mathrm{im}(\mathrm{id} \otimes f) \to M \otimes W$.

2. Let R and M be as in Problem 1 and assume additionally that M is a free R-module. Show that for every short exact sequence $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ the sequence
$$0 \to M \otimes U \xrightarrow{\mathrm{id}\otimes f} M \otimes V \xrightarrow{\mathrm{id}\otimes g} M \otimes W \to 0$$
is exact.

3. Let H, H', H'' and G be Abelian groups and $f : H \to H'$, $g : H' \to H''$ group homomorphisms. Show that f induces a well defined homomorphism $f_{\mathrm{Tor}} : \mathrm{Tor}(H, G) \to \mathrm{Tor}(H', G)$. Moreover show that $\mathrm{id}_{\mathrm{Tor}} = \mathrm{id}$, $(g \circ f)_{\mathrm{Tor}} = g_{\mathrm{Tor}} \circ f_{\mathrm{Tor}}$ and if f is an isomorphism then $(f^{-1})_{\mathrm{Tor}} = (f_{\mathrm{Tor}})^{-1}$.

4. Prove that the sequence in the universal coefficient theorem for homology is natural with respect to chain maps. That is, given a chain map $f : C_* \to D_*$ show that the diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(C) \otimes G & \longrightarrow & H_n(C; G) & \longrightarrow & \mathrm{Tor}(H_{n-1}(C), G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f_*} & & \downarrow & & \\ 0 & \longrightarrow & H_n(D) \otimes G & \longrightarrow & H_n(D; G) & \longrightarrow & \mathrm{Tor}(H_{n-1}(D), G) & \longrightarrow & 0 \end{array}$$
commutes.

   Remark: The statement also holds for the universal coefficient theorem for cohomology.

5. Let C_*, D_* be chain complexes of free Abelian groups and assume that $f : C_* \to D_*$ is a quasi-isomorphism, i.e. a chain map such that $f_* : H_*(C) \to H_*(D)$ is an isomorphism. Let G be an Abelian group. Prove the following statements using naturality of the sequences in the universal coefficient theorems.
   (a) $f \otimes \mathrm{id} : C_* \otimes G \to D_* \otimes G$ is a quasi-isomorphism.
   (b) $f^* : \mathrm{Hom}(D_*, G) \to \mathrm{Hom}(C_*, G)$ is a quasi-isomorphism.

6. Show that the splitting $H^n(X; G) \cong \mathrm{Ext}(H_{n-1}(X); G) \oplus \mathrm{Hom}(H_n(X); G)$ whose existence is asserted by the universal coefficient theorem for cohomology *cannot* be natural in X.

   Hint: Consider the map $\phi : \mathbb{R}P^2 \to S^2$ given by collapsing $\mathbb{R}P^1 \subset \mathbb{R}P^2$ to a point.

7. The Klein bottle K has $H_0(K; \mathbb{Z}) \cong \mathbb{Z}$, $H_1(K; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and all other homology groups vanish. Use this to compute the cohomology of K with coefficients in $\mathbb{Z}$ and the cohomology and homology with coefficients in $\mathbb{Z}_p$ for p prime.

8. Let X be a topological space and let $A, B \subset X$ be subsets. Denote by $S_k(A + B) \subset S_k(X)$ the subspace of chains which are sums of simplices entirely contained in A or B. Show that the quotient $S_k(X)/S_k(A + B)$ is free.