Prof. Paul Biran | Algebraic Topology II | FS 2019
ETH Zürich

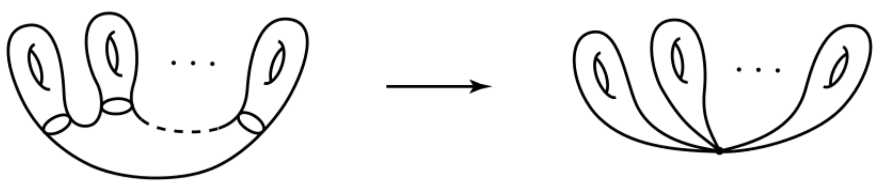
Problem set 3

1. Prove that there are natural ring isomorphisms
$$H^*(\coprod_\alpha X_\alpha; R) \to \prod_\alpha H^*(X_\alpha; R), \quad H^*(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha; R) \to \prod_\alpha H^*(X_\alpha, A_\alpha; R),$$
$$\widetilde{H}^*(\bigvee_\alpha X_\alpha; R) \cong \prod_\alpha \widetilde{H}^*(X_\alpha; R)$$
for every coefficient group R, where the product on the right is coordinatewise cup product. (Assume in the last case that the spaces are joined at points which are deformation retracts of neighbourhoods.)

2. Show that if $H_n(X;\mathbb{Z})$ is free for each n, then $H^*(X;R)$ and $H^*(X;\mathbb{Z}) \otimes R$ are isomorphic as rings.

3. Suppose that a space X can be covered by two acyclic open sets A and B. Using the cup product $H^k(X, A; R) \times H^\ell(X, B; R) \to H^{k+\ell}(X, A \cup B; R)$, show that all cup products of classes in $H^*(X;R)$ of positive dimensions vanish. Generalize to the situation that X can be covered by n acyclic open sets.

4. Compute the cup product structure on $H^*(\Sigma_g;\mathbb{Z})$ for the closed orientable surface Σ_g of genus g, assuming as known the cup product structure on $H^*(T^2;\mathbb{Z})$ and using the map $\pi : \Sigma_g \to \bigvee_g T^2$ depicted below.



5. Let X, Y be spaces and let $A \subset X$ be a subspace. Denote by $\delta : H^*(A;R) \to H^{*+1}(X, A; R)$ and $\delta' : H^*(A \times Y; R) \to H^{*+1}(X \times Y, A \times Y; R)$ the boundary maps from the LES for the pairs (X, A) and $(X \times Y, A \times Y)$. Prove that $\delta'(a \times b) = \delta(a) \times b$ for $a \in H^k(A;R)$ and $b \in H^\ell(Y;R)$, i.e. that the following diagram commutes:
$$\begin{array}{ccc} H^k(A;R) \times H^\ell(Y;R) & \xrightarrow{\delta\times\mathrm{id}} & H^{k+1}(X,A;R) \times H^\ell(Y;R) \\ \downarrow{\times} & & \downarrow{\times} \\ H^{k+\ell}(A \times Y;R) & \xrightarrow{\delta'} & H^{k+\ell+1}(X \times Y, A \times Y; R) \end{array}$$

6. Let $\mu_0 \in H^1(I, \partial I; R) \cong R$ a generator. Prove that the map $H^n(Y;R) \to H^{n+1}(I \times Y, \partial I \times Y; R)$, $\beta \mapsto \beta \times \mu_0$, is an isomorphism for every space Y and every $n \geq 0$. *Hint:* Consider the LES for the pair $(I \times Y, \partial I \times Y)$ and use the result of the previous problem!

7. Show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

8. Let X be the space obtained from $\mathbb{C}P^2$ by attaching a 3-cell by a map $S^2 \to \mathbb{C}P^1 \subset \mathbb{C}P^2$ of degree p, and let $Y = M(\mathbb{Z}_2, p) \vee S^4$, where $M(\mathbb{Z}_2, p)$ is S^2 with a 3-cell attached by a map $S^2 \to S^2$ of degree p (this is an example of a *Moore space*). Show that X and Y have isomorphic cohomology rings with $\mathbb{Z}$-coefficients, but not with $\mathbb{Z}_p$-coefficients.