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Solutions to problem set 2

1. Every element of $M \otimes W$ of the form $m \otimes w$ is in the image of $\mathrm{id} \otimes g$ because g is surjective; since every element of $M \otimes W$ is a sum of elements of this form, it follows that $\mathrm{id} \otimes g$ is surjective. By a similar argument one sees that $\mathrm{im}(\mathrm{id} \otimes f) \subseteq \ker(\mathrm{id} \otimes \mathrm{g})$.

   To prove $\ker(\mathrm{id} \otimes \mathrm{g}) \subseteq \mathrm{im}(\mathrm{id} \otimes f) =: I$, consider the map $\phi : M \otimes V/I \to M \otimes W$ induced by $\mathrm{id} \otimes g$, which is well-defined because $I \subseteq \ker(\mathrm{id} \otimes \mathrm{g})$. We now define a map $\psi : M \otimes W \to M \otimes V/I$ which is a left inverse for ϕ, i.e. such that $\psi \circ \phi = \mathrm{id}$; this implies injectivity of ϕ and hence that $\ker(\mathrm{id} \otimes g) \subseteq I$. To define ψ, consider first the map $M \times W \to M \otimes V/I$ defined as follows: It takes (m, w) to $[m \otimes v]$, where $v \in V$ is any element such that $g(v) = w$. This is well-defined and bilinear and hence descends to a map $\psi : M \otimes W \to M \otimes V/I$. We clearly have $\psi \circ \phi = \mathrm{id}$: That's obvious on elements of the form $[m \otimes v]$, and these generate.

2. In view of the previous problem, what is left to prove is the injectivity of $\mathrm{id} \otimes f$. Freeness of M means that it has a linearly independent generating set $\{m_i\}_{i \in I}$. Note that every element of $M \otimes U$ can be written as a sum $\sum_{i \in I} m_i \otimes u_i$ and that there is a well-defined map $M \otimes U \to \bigoplus_{i \in I} U$ taking such an element to $(u_i)_{i \in I}$. It follows that $(\mathrm{id} \otimes f)(\sum m_i \otimes u_i) = \sum m_i \otimes f(u_i) = 0$ implies $f(u_i) = 0$ for all i, hence $u_i = 0$ for all i by injectivity of f, and hence $\sum m_i \otimes u_i = 0$.

3. Let H, H' be Abelian groups with free resolutions $F \to H$, $F' \to H'$. By the free resolution lemma, we can extend any given group homomorphism $f : H \to H'$ to a chain map $\widetilde{f} : F \to F'$. Recall that by definition we have $\mathrm{Tor}(H, G) = H_1(F \otimes G)$ and $\mathrm{Tor}(H', G) = H_1(F' \otimes G)$, and so we define the action of $\mathrm{Tor}(-, G)$ on f by

   $$f_{\mathrm{Tor}} := (\widetilde{f} \otimes \mathrm{id})_* : H_1(F' \otimes G) \to H_1(F \otimes G).$$

   This is independent of the choice of lift $\widetilde{f}$ as that is unique up to chain homotopy. To see that this makes $\mathrm{Tor}(-, G)$ a functor, note that $\mathrm{id}_{\mathrm{Tor}} = \mathrm{id}$ because we can take as a lift of $\mathrm{id} : H \to H$ simply id of any free resolution of H. Moreover, $(fg)_{\mathrm{Tor}} = g_{\mathrm{Tor}} f_{\mathrm{Tor}}$, because if $\widetilde{f}$ lifts f and $\widetilde{g}$ lifts g, then $\widetilde{g}\widetilde{f}$ lifts gf.

   The case of $\mathrm{Ext}(-, G)$ is analogous. (Of course, these are are just special cases of how in general one constructs the action of derived functors on morphisms.)

4. We discuss the sequence $0 \to H_n(C) \to H_n(C \otimes G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0$ appearing in the universal coefficient theorem for homology. Recall that we constructed this as

   $$0 \to \mathrm{coker}(i_n \otimes \mathrm{id}) \to H_n(C; G) \to \ker(i_{n-1} \otimes \mathrm{id}) \to 0 \tag{1}$$

   with $i_n : B_n \to Z_n$ the inclusion map, and then noted that

   $$\mathrm{coker}(i_n \otimes \mathrm{id}) \cong H_n(C) \otimes G \quad \text{and} \quad \ker(i_{n-1} \otimes \mathrm{id}) \cong \mathrm{Tor}(H_{n-1}(C), G). \tag{2}$$

   It is clear that a chain map $\phi : C \to C'$ induces a morphism of short exact sequences between (1) and its counterpart for C' (just think about how we arrived at (1)). Moreover, one checks easily that under the identifications (2) and the corresponding ones for C', the outer maps in this morphism of SES are $\phi_* : H_n(C) \to H_n(C')$ and $(\phi_*)_{\mathrm{Tor}}$.

5. (a) Naturality of the short exact sequence in the universal coefficient theorem for homology says that the diagram

$$\begin{tikzcd} 0 \arrow[r] & H_n(C)\otimes G \arrow[r] \arrow[d, "f_*\otimes\mathrm{id}"] & H_n(C;G) \arrow[r] \arrow[d, "f_*"] & \mathrm{Tor}(H_{n-1}(C),G) \arrow[r] \arrow[d, "(f_*)_{\mathrm{Tor}}"] & 0 \\ 0 \arrow[r] & H_n(D)\otimes G \arrow[r] & H_n(D;G) \arrow[r] & \mathrm{Tor}(H_{n-1}(D),G) \arrow[r] & 0 \end{tikzcd}$$

commutes. The outer two maps are isomorphisms because $f_* : H_*(C) \to H_*(D)$ is an isomorphism by assumption and by functoriality of $\mathrm{Tor}(-,G)$. Hence $f_* : H_*(C;G) \to H_*(D;G)$ is an isomorphism by the 5-lemma.

(b) Same argument as in (a) using the universal coefficient theorem for cohomology.

6. Consider the diagram

$$\begin{tikzcd} H^2(S^2;G) \arrow[r] \arrow[d, "\phi^*"'] & \mathrm{Ext}(H_1(S^2),G)\oplus\mathrm{Hom}(H_2(S^2),G) \arrow[d, "(\phi_*)^{\mathrm{Ext}}\oplus(\phi_*)^*"] \\ H^2(\mathbb{R}P^2;G) \arrow[r] & \mathrm{Ext}(H_1(\mathbb{R}P^2),G)\oplus\mathrm{Hom}(H_2(\mathbb{R}P^2),G) \end{tikzcd}$$

Note that we have $\mathrm{Ext}(H_1(S^2),G) = 0$ and $\mathrm{Hom}(H_2(\mathbb{R}P^2),G) = 0$ because $H_1(S^2) = 0$, $H_2(\mathbb{R}P^2) = 0$, and hence the map on the right vanishes for every Abelian group G. If the splitting were natural, the map $\phi^* : H^2(S^2;G) \to H^2(\mathbb{R}P^2;G)$ would consequently also have to vanish for every G.

We will show, in contrast, that $\phi^* : H^2(S^2;\mathbb{Z}_2) \to H^2(\mathbb{R}P^2;\mathbb{Z}_2)$ is an isomorphism. To see this, note that $\phi : \mathbb{R}P^2 \to S^2$ is a cellular map with respect to the usual CW complex structures of $\mathbb{R}P^2$ (with one cell in each degree $0,1,2$) and S^2 (with one cell in degree 0 and one in degree 2). The map induced by ϕ on cellular chains takes the generator corresponding to the unique 2-cell of $\mathbb{R}P^2$ to the generator corresponding to the unique 2-cell of S^2 (recall the description of this map!). Dualizing, this implies that the map induced by ϕ on the cellular cochain complexes with coefficients in $\mathbb{Z}_2$ looks as follows:

$$\begin{tikzcd} 0 & \mathbb{Z}_2 \arrow[l] & \mathbb{Z}_2 \arrow[l, "0"'] & \mathbb{Z}_2 \arrow[l, "0"'] & 0 \arrow[l] \\ 0 & \mathbb{Z}_2 \arrow[l] \arrow[u, "\cong"] & 0 \arrow[l] \arrow[u] & \mathbb{Z}_2 \arrow[l] \arrow[u, "\cong"] & 0 \arrow[l] \end{tikzcd}$$

In particular, the induced map $H^2(S^2;\mathbb{Z}_2) \to H^2(\mathbb{R}P^2;\mathbb{Z}_2)$ is an isomorphism.

7. The universal coefficient theorem for homology tells us that there is a splitting

$$H_n(K;G) \cong (H_n(K)\otimes G)\oplus \mathrm{Tor}(H_{n-1}(K),G)$$

for every Abelian group G. We have $H_0(K)\otimes\mathbb{Z}_p = \mathbb{Z}_p$ and $H_1(K)\otimes\mathbb{Z}_p = \mathbb{Z}_p \oplus (\mathbb{Z}_2\otimes\mathbb{Z}_p)$; note that $\mathbb{Z}_2\otimes\mathbb{Z}_2 = \mathbb{Z}_2$ and $\mathbb{Z}_2\otimes\mathbb{Z}_p = 0$ for odd p (which doesn't have to be prime for that; in general, $\mathbb{Z}_q\otimes\mathbb{Z}_{q'} = 0$ if q,q' are coprime, as $1 = qm + q'm'$ for certain $m,m' \in \mathbb{Z}$, from which it follows that $1\otimes 1 = 0$ in $\mathbb{Z}_q\otimes\mathbb{Z}_{q'}$). Moreover, $\mathrm{Tor}(H_0(K),\mathbb{Z}_p) = 0$ as $H_0(K)$ is free and $\mathrm{Tor}(H_1(K),\mathbb{Z}_p) = \mathrm{Tor}(\mathbb{Z}_2,\mathbb{Z}_p) = \ker(\mathbb{Z}_p \xrightarrow{2} \mathbb{Z}_p)$, which is $\mathbb{Z}_2$ for $p=2$ and 0 if p is odd. Combining all that, we obtain

$$H_0(K;\mathbb{Z}_2) = \mathbb{Z}_2, \quad H_1(K;\mathbb{Z}_2) = \mathbb{Z}_2\oplus\mathbb{Z}_2, \quad H_2(K;\mathbb{Z}_2) = \mathbb{Z}_2$$

and

$$H_0(K;\mathbb{Z}_p) = \mathbb{Z}_p, \quad H_1(K;\mathbb{Z}_p) = \mathbb{Z}_p, \quad H_2(K;\mathbb{Z}_p) = 0$$

for p odd. All other groups vanish.

From the universal coefficients theorem for cohomology, we obtain a splitting

$$H^n(K;G) \cong \mathrm{Ext}(H_{n-1}(K), G) \oplus \mathrm{Hom}(H_n(K); G)$$

for every Abelian group G. We have $\mathrm{Ext}(H_0(K), G) = 0$ as $H_0(K)$ is free and $\mathrm{Ext}(H_1(K); G) = \mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G$, which is $\mathbb{Z}_2$ for $G = \mathbb{Z}$ or $G = \mathbb{Z}_2$ and 0 for $G = \mathbb{Z}_p$ with p odd. Moreover, $\mathrm{Hom}(H_0(K); G) = G$, and $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$ implies that

$$\mathrm{Hom}(H_1(K); G) = \begin{cases} \mathbb{Z} \otimes \mathbb{Z}_2, & G = \mathbb{Z} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & G = \mathbb{Z}_2 \\ \mathbb{Z}_p, & G = \mathbb{Z}_p \text{ with } p \text{ odd} \end{cases}$$

It follows that

$$H^0(K;\mathbb{Z}) = \mathbb{Z}, \quad H^1(K;\mathbb{Z}) = \mathbb{Z}, \quad H^2(K;\mathbb{Z}) = \mathbb{Z}_2,$$
$$H^0(K;\mathbb{Z}_2) = \mathbb{Z}_2, \quad H^1(K;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H^2(K;\mathbb{Z}_2) = \mathbb{Z}_2$$

and

$$H^0(K;\mathbb{Z}_p) = \mathbb{Z}_p, \quad H^1(K;\mathbb{Z}_p) = \mathbb{Z}_p, \quad H^2(K;\mathbb{Z}_p) = 0$$

for p odd. Again all other groups vanish.

8. $S_k(X)$ splits as $S_k(X) = S_k(A+B) \oplus S_k^{\perp}(A+B)$, where the second summand is generated by all simplices neither contained in A nor in B. Hence the quotient $S_k(X)/S_k(A+B)$ is isomorphic to $S_k^{\perp}(A+B)$, which is free.