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Algebraic Topology II

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# Solutions to problem set 3

**Notation.** We often omit the coefficient groups or rings from the notation, but they should always be clear from the context.

1. Notice that if $\phi \in S^k(\coprod_\alpha X_\alpha)$ is a cocycle that is non-zero only on chains in $X_\alpha$ for some $\alpha$ and $\psi \in S^k(\coprod_\alpha X_\alpha)$ is a cocycle that is non-zero only on chains in $X_\beta$ for some $\beta$ then $\phi \cup \psi$ is zero if $\alpha \neq \beta$. Moreover, if $\alpha = \beta$ then $\phi \cup \psi$ vanishes an all chains not contained in $X_\alpha$. The relative case follows analogously.

   The last isomorphism follows from the fact that $\tilde{H}^n(X;R) \cong H^n(X, x_0; R)$.

2. We have $H^n(X;R) \cong \mathrm{Hom}(H_n(X), R) \cong \mathrm{Hom}(H_n(X), \mathbb{Z}) \otimes R \cong H^n(X) \otimes R$ as Abelian groups, using the fact that $H_n(X)$ is free for all $n$ and universal coefficient theorem for cohomology. Given a cocycle $\phi \in S^n(X)$ and $r \in R$, this isomorphism identifies the class $[\phi] \otimes r \in H^n(X) \otimes R$ with the class in $H^n(X;R)$ represented by the cocycle in $S^n(X;R)$ that takes a chain $\sigma \in S_n(X)$ to $\phi(\sigma) r$. That this respects the ring structures is immediate from the definitions.

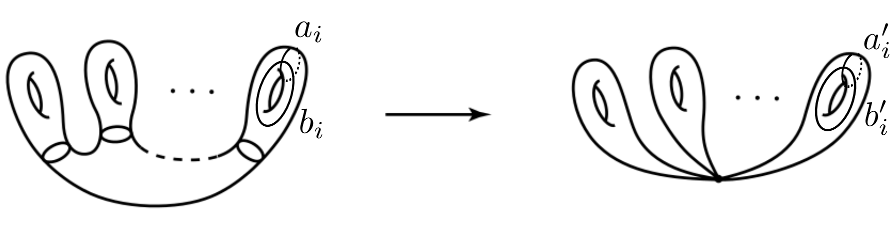
3. Consider the commutative diagram

$$\begin{array}{ccc} H^k(X,A) \times H^\ell(X,B) & \xrightarrow{\smile} & H^{k+\ell}(X, A \cup B) \\ \cong \downarrow & & \downarrow \\ H^k(X) \times H^\ell(X) & \xrightarrow{\smile} & H^{k+\ell}(X) \end{array}$$

   The left vertical map is an isomorphism because $A, B$ are acyclic and $k, \ell > 0$, as one sees by looking at the LES for the pairs $(X, A)$ and $(X, B)$; moreover, we have $H^{k+\ell}(X, A \cup B) = 0$ as $A \cup B = X$ by assumption. Combining these facts, it follows that the lower horizontal map vanishes.

   If $X = A_1 \cup \cdots \cup A_n$ with acyclic open sets $A_i$, it follows in a similar way that all $n$-fold cup products of classes in $H^*(X)$ of positive dimensions vanish.

4. Denote by $a_i, b_i,\ i = 1, \ldots, g$, the standard basis elements of $H_1(\Sigma_g)$ and by $a_i', b_i'$ the standard basis elements of $H_1(X)$, where $X = \bigvee_g T^2$ (as indicated in the figure). Moreover, let $\alpha_i, \beta_i$ be the elements of the dual basis of $H^1(\Sigma_g) \cong \mathrm{Hom}(H_1(\Sigma_g), \mathbb{Z})$, and $\alpha_i', \beta_i'$ the elements of the dual basis of $H^1(X) \cong \mathrm{Hom}(H_1(X), \mathbb{Z})$. We have $\pi_* a_i = a_i'$, $\pi_* b_i = b_i'$ as




   $\pi : \Sigma_g \to X$ takes curves representing the classes on $T^2$ to curves representing the classes on $X$. Dualizing, it follows that $\pi^* \alpha_i' = \alpha_i$ and $\pi^* \beta_i' = \beta_i$.

   The isomorphism $\iota_1^* \oplus \cdots \oplus \iota_g^* : H^*(X) \to \bigoplus_i H^*(T^2)$ induced by the inclusion maps $\iota_i : T^2 \to X$ is an isomorphism of rings, where the ring structure on the right is given by componentwise multiplication (see Problem 1). It follows that $\alpha'_i \smile \alpha'_j = \alpha'_i \smile \beta'_j = \beta'_i \smile \beta'_j = 0$ for $i \neq j$ because these classes live in different summands. Moreover, $\alpha'_i \smile \alpha'_i = 0 = \beta'_i \smile \beta'_i$ and $\alpha'_i \smile \beta'_i = (0, \ldots, 0, \gamma_{T^2}, 0, \ldots, 0) \in H^2(X)$ using that the cup product structure on $H^*(T^2)$ is known and denoting by $\gamma_{T^2}$ a generator of $H^2(T^2)$ (for instance, $\iota_i^*(\alpha'_i \smile \beta'_i) = \alpha \smile \beta = \gamma_{T^2} \in H^2(T^2)$ where now $\alpha, \beta$ denote generators of $H^1(T^2)$).

Denote by $[T^2]$ the generator of $H_2(T^2)$ dual to $\gamma_{T^2}$ (note $H^2(T^2) \cong \mathrm{Hom}(H_2(T^2), \mathbb{Z})$) and by $[\Sigma_g]$ the generator of $H_2(\Sigma_g)$ such that $\pi_*([\Sigma_g]) = ([T^2], \ldots, [T^2])$ (one can see that such a generator exists using e.g. cellular homology). Then $(\alpha_i \smile \beta_i)[\Sigma_g] = (\pi^*\alpha'_i \smile \pi^*\beta'_i)[\Sigma_g] = (\alpha'_i \smile \beta'_i)(\pi_*[\Sigma_g]) = (\alpha'_i \smile \beta'_i)([T^2], \ldots, [T^2]) = 1$, and hence $\alpha_i \smile \beta_i = \gamma_{\Sigma_g}$, the generator of $H^2(\Sigma_g) \cong \mathrm{Hom}(H_2(\Sigma_g), \mathbb{Z})$ dual to $[\Sigma_g]$; by skew-commutativity, we have $\beta_i \smile \alpha_i = -\alpha_i \smile \beta_i = -\gamma_{\Sigma_g}$. All other cup products between the basis elements of $H^1(\Sigma_g)$ vanish by the description above.

5. Let $\alpha \in S^k(A)$ and $\beta \in S^\ell(Y)$ be cocycles representing $a$ and $b$. Recall that $\delta a$ is represented by $\delta\overline{\alpha}$, where $\overline{\alpha} \in S^k(X)$ is any extension of $\alpha$ to a cochain in $X$ and where the second $\delta$ is the coboundary homomorphism $S^*(X) \to S^{*+1}(X)$. Denote by $p_1 : (X \times Y, A \times Y) \to (X, A)$ and $p_2 : X \times Y \to Y$ the projections. With this notation, $\delta(a) \times b$ is represented by the relative cocycle $p_1^*(\delta\overline{\alpha}) \smile p_2^*(\beta)$. On the other hand, $\delta'(a \times b)$ is represented by the relative cocycle $\delta'(p_1^*\overline{\alpha} \smile p_2^*\beta) = p_1^*(\delta\overline{\alpha}) \smile p_2^*(\beta) \pm p_1^*(\overline{\alpha}) \smile p_2^*(\delta\beta) = p_1^*(\delta\overline{\alpha}) \smile p_2^*(\beta)$; here we use that $p_1^*\overline{\alpha} \smile p_2^*\beta \in S^{k+\ell}(X \times Y)$ is an extension of $p_1^*\alpha \smile p_2^*\beta \in S^{k+\ell}(A \times Y)$ and the fact that $\beta \in S^\ell(Y)$ is a cocycle.

6. Consider the LES in cohomology for the pair $(I \times Y, \partial I \times Y)$. Since the maps $i^* : H^n(I \times Y) \to H^n(Y \times \partial I)$ are injective (given by $i^*(a) = (a, a)$ in the obvious identifications $H^*(I \times Y) \cong H^*(Y)$ and $H^*(\partial I \times Y) \cong H^*(Y) \oplus H^*(Y)$), the LES splits into SESs of the form
$$0 \to H^n(I \times Y) \xrightarrow{i^*} H^n(\partial I \times Y) \xrightarrow{\delta'} H^{n+1}(I \times Y, \partial I \times Y) \to 0$$
which split as $i^*$ has a left inverse (e.g. $(a, b) \mapsto a$). Define $1_0 \in H^0(\partial I)$ to be the class represented by the cocycle $\varphi_0$ with $\varphi_0(0) = 1$ and $\varphi_0(1) = 0$, and similarly define $1_1 \in H^0(\partial I)$. One checks easily that the composition $H^n(Y) \cong H^n(I \times Y) \xrightarrow{i^*} H^n(\partial I \times Y)$ is given by $b \mapsto 1_0 \times b + 1_1 \times b$, so the subspace $Q := \{1_0 \times b \mid b \in H^n(Y)\} \subset H^n(\partial I \times Y)$ is complementary to the image of $i^*$. Hence $\delta'|_Q : Q \to H^{n+1}(I \times Y, \partial I \times Y)$ is an isomorphism; since by the previous problem we have $\delta'(1_0 \times b) = \delta(1_0) \times b$, it follows that $H^n(Y) \to H^{n+1}(I \times Y, \partial I \times Y)$, $b \mapsto \delta(1_0) \times b$, is an isomorphism. This is what we need to prove in case $\mu_0 = \delta(1_0)$; any other generator $\mu_0 \in H^1(I, \partial I)$ is of the form $\mu_0 = \delta(1_0) \cdot r$ for some invertible $r \in R$, and thus $b \mapsto \mu_0 \times b$ is also an isomorphism in this case.

7. The $\mathbb{Z}_2$-cohomology ring of $\mathbb{R}P^3$ is $H^*(\mathbb{R}P^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^4)$ with $|\alpha| = 1$, whereas that of $\mathbb{R}P^2 \vee S^3$ is $H^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[\beta]/(\beta^3) \oplus \mathbb{Z}_2[\gamma]/(\gamma^2)$ with $|\beta| = 1$ and $|\gamma| = 3$ using the result of the Problem 1. These are isomorphic as $\mathbb{Z}_2$-vector spaces but not as rings (e.g. because the generator of $H^1$ squares to zero in the second but not in the first case).

8. Using cellular homology, one computes
$$H_i(X; \mathbb{Z}), H_i(Y; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}_p & i = 2 \\ 0 & i = 3 \\ \mathbb{Z} & i = 4 \end{cases}$$

Using the universal coefficients theorem for cohomology it follows that

$$H^i(X;\mathbb{Z}), H^i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=1 \\ 0 & i=2 \\ \mathbb{Z}_p & i=3 \\ \mathbb{Z} & i=4 \end{cases} \quad \text{and} \quad H^i(X;\mathbb{Z}_p), H^i(Y;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & i=0 \\ 0 & i=1 \\ \mathbb{Z}_p & i=2 \\ \mathbb{Z}_p & i=3 \\ \mathbb{Z}_p & i=4 \end{cases}$$

The cohomology rings with $\mathbb{Z}$-coefficients are clearly isomorphic (the only non-vanishing cup products are multiplication with multiples of the unit in $H^0$).

We now compare the cohomology rings with $\mathbb{Z}_p$-coefficients: Let $\alpha \in H^2(X;\mathbb{Z}_p)$ be a generator; using the cellular description of induced maps, one sees that the map $i^* : H^2(X;\mathbb{Z}_p) \to H^2(\mathbb{C}P^2;\mathbb{Z}_p)$ induced by the inclusion $i : \mathbb{C}P^2 \to X$ takes $\alpha$ to a generator $f^*(\alpha)$ of $H^2(\mathbb{C}P^2;\mathbb{Z}_p)$. Using the known description of the ring structure of $H^*(\mathbb{C}P^2)$ and the fact that $H^*(\mathbb{C}P^2;\mathbb{Z}_p) \cong H^*(\mathbb{C}P^2;\mathbb{Z}) \otimes \mathbb{Z}_p$ as rings (see Problem 2), it follows that $f^*(\alpha \smile \alpha) = f^*(\alpha) \smile f^*(\alpha) \neq 0$.

In contrast to that, if $\beta \in H^2(Y;\mathbb{Z}_p)$ is a generator, we have $\beta \smile \beta = 0$. To see that, recall the ring isomorphism $\widetilde{H}^*(Y;\mathbb{Z}_p) \cong \widetilde{H}^*(M(\mathbb{Z}_p;2);\mathbb{Z}_p) \oplus \widetilde{H}^*(S^4;\mathbb{Z}_p)$ from Problem 1. The class $\beta$ lives in the first factor of this splitting which vanishes in dimension 4.