Solutions to problem set 5

1. Given a closed manifold M, we set $M' := M \setminus \mathrm{pt}$. Using the Mayer-Vietoris sequence for the cover of M given by M' and a ball, one sees that $H_i(M') = H_i(M)$ for $i < n-1$. We also know that $H_n(M') = 0$ as M' is non-compact, and hence the top end of the MV sequence is

$$0 \to H_n(M) \to H_{n-1}(S^{n-1}) \to H_{n-1}(M') \to H_{n-1}(M) \to 0 \tag{1}$$

If M is orientable ($\Leftrightarrow$ $H_n(M) \cong \mathbb{Z}$), the first map is an isomorphism so that we get $H_{n-1}(M') = H_{n-1}(M)$ in this case, whereas if M is non-orientable ($\Leftrightarrow H_n(M) = 0$), we end up with a short exact sequence

$$0 \to H_{n-1}(S^{n-1}) \to H_{n-1}(M') \to H_{n-1}(M) \to 0 \tag{2}$$

Note in particular that $b_{n-1}(M') = b_{n-1}(M) + 1$ in this case.

To compute $H_*(M_1 \# M_2)$, consider the cover of $M_1 \# M_2$ given by two sets $A_1 \approx M_1'$, $A_2 \approx M_2'$ with $A_1 \cap A_2 \simeq S^{n-1}$. From the resulting Mayer-Vietoris sequence we see immediately that $H_i(M_1 \# M_2) \cong H_i(M_1') \oplus H_i(M_2') \cong H_i(M_1) \oplus H_i(M_2)$ for $0 < i < n-1$. The top end of the MV sequence looks as follows:

$$0 \to H_n(M_1 \# M_2) \to H_{n-1}(S^{n-1}) \xrightarrow{\phi} H_{n-1}(M_1') \oplus H_{n-1}(M_2') \to H_{n-1}(M_1 \# M_2) \to 0 \tag{3}$$

Writing $\phi = (\phi_1, \phi_2)$, note that the maps $\phi_i : H_{n-1}(S^{n-1}) \to H_{n-1}(M_i')$ are precisely those also appearing in (1) with $M = M_i$. So if both M_1 and M_2 are orientable, ϕ vanishes and we obtain $H_{n-1}(M_1 \# M_2) \cong H_{n-1}(M_1) \oplus H_{n-1}(M_2)$; we also see that $H_n(M_1 \# M_2) \cong \mathbb{Z}$, so $M_1 \# M_2$ is orientable. If at least one of the M_i is non-orientable, ϕ is injective and we obtain from (3) a SES

$$0 \to H_{n-1}(S^{n-1}) \xrightarrow{\phi} H_{n-1}(M_1') \oplus H_{n-1}(M_2') \to H_{n-1}(M_1 \# M_2) \to 0 \tag{4}$$

(and $H_n(M_1 \# M_2) = 0$, so $M_1 \# M_2$ is non-orientable). If precisely one of the M_i (say M_2) is non-orientable, this yields $H_{n-1}(M_1 \# M_2) \cong \operatorname{coker} \phi \cong H_{n-1}(M_1) \oplus H_{n-1}(M_2)$ because $\phi_1 = 0$ and $\operatorname{coker} \phi_2 \cong H_{n-1}(M_2)$ using the SES (2) with $M = M_2$. If both M_1 and M_2 are non-orientable, we obtain $1 - b_{n-1}(M_1') - b_{n-1}(M_2') + b_{n-1}(M_1 \# M_2) = 0$ by exactness of (4), hence $b_{n-1}(M_1 \# M_2) = b_{n-1}(M_1) + b_{n-1}(M_2) + 1$ (using $b_{n-1}(M_i') = b_{n-1}(M_i) + 1$). Combining this with what we know from Problem 1, we conclude that $H_{n-1}(M_1 \# M_2)$ is obtained by replacing one $\mathbb{Z}_2$-summand in $H_{n-1}(M_1) \oplus H_{n-1}(M_2)$ by a $\mathbb{Z}$-summand.

2. Poincaré duality tells us that $H_{n-1}(M) \cong H^{n+1}(M)$, and $H^{n+1}(M) \cong \operatorname{Hom}(H_{n+1}, \mathbb{Z}) \oplus \operatorname{Ext}(H_n(M), Z)$ by the universal coefficient theorem. Recall that $\operatorname{Ext}(G, \mathbb{Z})$ is isomorphic to the torsion subgroup of G for any finitely generated Abelian group G (which the $H_i(M)$ are as M is a compact manifold). So if $H_n(M)$ has torsion, then also $H^{n+1}(M)$ and $H_{n-1}(M)$ have torsion.

3. Consider the maps $p : (S^2 \times S^8) \# (S^4 \times S^6) \to S^2 \times S^8$ and $q : (S^2 \times S^8) \# (S^4 \times S^6) \to S^4 \times S^6$ given by collapsing one of the two summands. It follows from the result of problem 1 that $p^* \oplus q^* : H^i(S^2 \times S^8) \oplus H^i(S^4 \times S^6) \to H^i(S^2 \times S^8 \# S^4 \times S^6)$ is an isomorphism in degrees $0 < i < 10$ (check that our $p^* \oplus q^*$ is just the dual of the isomorphism $H_i(S^2 \times S^8 \# S^4 \times S^6) \cong$

$H_i(S^4 \times S^6) \oplus H_i(S^2 \# S^8)$ from problem 1!). The fact that this is a ring homomorphism shows that the only non-trivial cup products are those between elements of complementary degrees, i.e. those forced by Poincaré duality.

4. Recall from class that every odd-dimensional manifold has vanishing Euler characteristic, so $0 = \chi(M) = b_0 - b_1 + b_2 - b_3$. We have $b_0 = 1$, and $b_3 = 0$ since M is non-orientable. Hence $b_1 > 0$ and thus $H_1(M)$ is infinite.

5. Denoting by $\alpha \in H^2(\mathbb{C}P^n)$ a generator, α^n generates $H^{2n}(\mathbb{C}P^n)$ and we have $\alpha^n \frown [\mathbb{C}P^n] = 1$ for one choice of fundamental class $[\mathbb{C}P^n] \in H_{2n}(\mathbb{C}P^n)$. Take $k \in \mathbb{Z}$ such that $f^*(\alpha) = k\alpha$. Then $f^*(\alpha^n) = (f^*(\alpha))^n = k^n\alpha^n$ and hence $\alpha^n \frown f_*([\mathbb{C}P^n]) = f^*(\alpha^n) \frown [\mathbb{C}P^n] = k^n\alpha^n \frown [\mathbb{C}P^n] = k^n$. It follows that $f_*([\mathbb{C}P^n]) = k^n[\mathbb{C}P^n]$, i.e. f has degree k^n.

6. The map $H^n(S^n) \oplus H^n(S^n) \cong H^n(S^n \times S^n)$, $(k\alpha, \ell\alpha) \mapsto k(\alpha \times 1) + \ell(1 \times \alpha) = ku + \ell v$, is an isomorphism by Künneth. Being a product of orientable manifolds, $S^n \times S^n$ is orientable and thus the cup product pairing is non-singular, which implies that there exists some $u' \in H^n(S^n \times S^n)$ such that $u \smile u'$ generates $H^{2n}(S^n \times S^n)$. Since $u \smile u = (p_0^*\alpha \smile p_1^*1) \smile (p_0^*\alpha \smile p_1^*1) = p_0^*(\alpha \smile \alpha) \smile p_1^*(1 \smile 1) = 0$ as $\alpha \smile \alpha = 0$ (where $p_i : S^n \times S^n \to S^n$ denote the projections to the factors), we can choose $u' = v$.

   So $u \smile v$ generates $H^{2n}(S^n \times S^n)$, and thus by Poincaré duality we know that $(u \smile v) \frown [S^n \times S^n] = \pm 1$, where $[S^n \times S^n]$ is a fundamental class. It follows that $f^*(u \smile v) = \pm u \smile v$, using that $f^*(u \smile v) \frown [S^n \times S^n] = (u \smile v) \frown f_*[S^n \times S^n] = \pm 1$ by the assumption that $\deg f = \pm 1$. Note that $u \smile v = v \smile u$ a n is even; using that and $u \smile u = 0 = v \smile v$, we obtain

$$f^*(u \smile v) = f^*(u) \smile f^*(v) = (au + bv) \smile (cu + dv) = (ad + bc)u \smile v,$$

$$f^*(u \smile u) = (au + bv) \smile (au + bv) = 2ab(u \smile v) = 0,$$

$$f^*(v \smile v) = (cu + dv) \smile (cu + dv) = 2cd(u \smile v) = 0.$$

   So $ad + bc = \pm 1$ and $ab = 0 = cd$, which is equivalent to what we need to prove.

7. Let $[S^n] \in H_n(S^n; \mathbb{Q})$ and $[M] \in H_n(M; \mathbb{Q})$ be fundamental classes. Writing $k = \deg f$, we have $f_*[S^n] = k[M]$. Fix now $0 < i < n$ and let $\sigma \in H_i(M; \mathbb{Q})$ be any class with Poincaré dual $\alpha \in H^{n-i}(M; \mathbb{Q})$, i.e. $\sigma = \alpha \frown [M]$. Then $k\sigma = k\alpha \frown [M] = \alpha \frown f_*[S^n] = f^*\alpha \frown [S^n] = 0$ because $f^*\alpha \in H^{n-i}(S^n)$ vanishes for degree reasons. Since we are working over $\mathbb{Q}$, it follows that $\sigma = 0$. So $H_i(M; \mathbb{Q}) = 0$ for $0 < i < n$ and we conclude $H_*(M; \mathbb{Q}) \cong H_*(S^n; \mathbb{Q})$ (in degrees 0 and n this is clear as M is closed connected orientable). If we replace $\mathbb{Q}$ by $\mathbb{Z}$, the same argument shows that every $\sigma \in H_k(M)$ is k-torsion; in particular, if we assume $k = \pm 1$ we obtain $H_i(M) = 0$ for $0 < i < n$ and thus $H_*(M) \cong H_*(S^n)$.

8. Suppose that $H_i(M) \neq 0$ for some $0 < i < n$. If $H_i(M)$ contains a non-torsion element σ, consider its Poincaré dual $\alpha \in H^{n-i}(M)$; by the non-singularity of the cup product pairing there exists some $\beta \in H^i(M)$ such that $\alpha \smile \beta$ generates $H^n(M)$. Otherwise there is a non-zero σ that is p-torsion for some prime p and the universal coefficient theorem for homology implies that there exists a non-zero element $\sigma' \in H_i(M; \mathbb{Z}_p)$ with Poincaré dual $\alpha' \in H^{n-i}(M; \mathbb{Z}_p)$; by the non-singularity of the cup product pairing (now over the field $\mathbb{Z}_p$) there exists some $\beta' \in H^i(M; \mathbb{Z}_p)$ such that $\alpha' \smile \beta'$ generates $H^n(M; \mathbb{Z}_p) \cong \mathbb{Z}_p$.

   On the other hand, the assumption that $M = U \cup V$ with acyclic U, V implies that all cup products of classes of positive degree in $H^*(M)$ resp. $H^*(M; \mathbb{Z}_p)$ vanish (see Problem 3/3).

   This contradiction shows that $H_i(M) = 0$ for $0 < i < n$ and hence $H_*(M) \cong H_*(S^n)$.