D-MATH
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Algebraic Geometry

FS 2019

Exercise Sheet 1

Classical Varieties

Let K be an algebraically closed field. All algebraic sets below are defined over K, unless specified otherwise.

1. Describe the Zariski topology on $Z(XY) \subset \mathbb{A}^2$.

2. Assume that $\mathrm{char}(K) \neq 2, 3$. Show that the polynomial $Y^2 - X^3 - X \in K[X, Y]$ is irreducible. Describe the Zariski topology on $Z(Y^2 - X^3 - X) \subset \mathbb{A}^2$.

3. Let Y be the algebraic set of $\mathbb{A}^3_K$ defined by the two polynomials $X^2 - YZ$ and $XZ - X$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

4. Let $A \subset \mathbb{A}^n$ and $B \subset \mathbb{A}^m$ be two algebraic sets. Prove that their product set $A \times B \subset \mathbb{A}^{n+m}$ is algebraic, too.

5. Let K be a field. An algebraic subset of $K^{n^2} = \mathcal{M}_{n\times n}(K)$ that is a subgroup of $\mathrm{GL}_n(K)$ is called a *linear algebraic group.*

   (a) Show that the following are linear algebraic group

   $$\mathrm{SL}_n(K), \quad \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \; a^2 + b^2 = 1 \right\}, \quad \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}.$$

   (b) Show that if $H \subset \mathrm{SL}_n(K)$ is any subgroup then the Zariski closure is a linear algebraic group.

   (c) Let $K = \mathbb{C}$ and $n = 2$. Compute the Zariski closure of $H = \mathrm{SL}_2(\mathbb{Z})$.

6. Determine the Zariski closure for the following subsets:

   (a) $\{(x, \sin(x)) \mid x \in \mathbb{C}\} \subset \mathbb{A}^2_{\mathbb{C}}$

   (b) $\{(a^2 - b^2, 2ab, a^2 + b^2) \mid a, b \in \mathbb{Z}\} \subset \mathbb{A}^3_{\mathbb{C}}$.

7. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism. This shows that not every morphism whose underlying map of topological spaces is a homeomorphism needs to be an isomorphism.

8. Let $Y \subset \mathbb{A}^3$ be the set $Y := \{(t, t^2, t^3) \mid t \in K\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$ and prove that the coordinate ring $\mathcal{O}(Y)$ is isomorphic to a polynomial ring in one variable over K.

9. (a) Show that there are not non-constant rational functions $f, g \in \mathbb{C}(X)$, such that
$$f^2 = g^3 - g.$$
   (b) Show that there exist non-constant rational functions $f, g \in \mathbb{C}(X)$, such that
$$f^2 = g^3.$$

10. Let $Y \subset \mathbb{A}^3_K$ be the curve given parametrically by $x = t^3$, $y = t^4$, $z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $K[X, Y, Z]$ which cannot be generated by 2 elements. We say that Y is *not a local complete intersection.* Proceed as follows:

   (a) Show that the closed subsets of Y are given by
$$\{Y\} \cup \{C \subset Y : |C| < \infty\}.$$
   Conclude that Y is irreducible and that $\dim Y = 1$.

   (b) Let $f \in\subset K[X, Y, Z]$, then we can write
$$f(X, Y, Z) = \sum_{n_1, n_2, n_3} c_{n_1, n_2, n_3} X^{n_1} Y^{n_2} Z^{n_3}.$$
   Show that if $f \in I(Y)$, then
$$\sum_{\substack{n_1, n_2, n_3 \\ 3n_1 + 4n_2 + 5n_3 = n}} c_{n_1, n_2, n_3} = 0,$$
   for any $n \geqslant 0$. Use this to show that
$$Y^2 - XZ, \quad X^3 - YZ, \quad X^2Y - Z^2 \in I(Y),$$
   and deduce that Y is an affine variety.

   (c) Conclude the exercise.

11. *The Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$ to $(\dots, a_i b_j, \dots)$ in lexicographic order, where $N = rs + r + s$. Show that ψ is well-defined and injective. It is called the *Segre embedding.* Prove that the image of ψ is a projective algebraic set in $\mathbb{P}^N$.

12. Consider the surface Q (i.e. variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw$.

(a) Show that Q is equal to the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

(b) Show that Q contains two families of lines (i.e. linear varieties of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbb{P}^1$, with the property that if $L_t \neq L_u$, then $L_t \cap L_u = \varnothing$; if $M_t \neq M_u$ then $M_t \cap M_u = \varnothing$ and for all t, u we have $L_t \cap M_u =$ one point.

(c) Show that Q contains other curves besides these lines and deduce that the Zariski topology on Q is not homeomorphic via the Segre embedding to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$.

13. Let $n, d > 0$ be integers. We denote by $M_0, \ldots, M_N$ all monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, where $N := \binom{n+d}{n} - 1$. We define the d-uple embedding as the map $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ sending the point $a = (a_0, \ldots, a_n)$ to the point $(M_0(a), \ldots, M_N(a))$. Show that the d-uple embedding of $\mathbb{P}^n$ is an isomorphism onto its image.

[Hint: Look at the inverse map.]