Exercise Sheet 2

Classical Varietes, Rational Maps, Blowups, Spectrum

Let K be an algebraically closed field. All algebraic sets and varieties below are defined over K, unless specified otherwise.

1. Consider the set $M := \mathrm{Mat}_{m,n}(K)$ of $m \times n$-matrices. It can be identified with the affine algebraic variety $\mathbb{A}^{nm}$. Determine if S is open/closed/dense in M:

   (a) $S := \{A \in M \mid A^t A \text{ has an eigenvalue } 1\}$

   (b) $S := \{A \in M \mid \mathrm{rank}(A) = \min\{m, n\}\}$

   (c) for $m = n$, $S := \{A \in M \mid A \text{ is diagonalisable}\}$

2. Show that a commutative K-algebra, A, is of the form $\mathcal{O}(Y)$ for some algebraic set Y if and only if it is finitely generated and it does not contains non-zero nilpotent elements.

3. Let $n \geqslant 1$ be an integer and $1 \leqslant k \leqslant n$. Let

   $$G_{n,k} := \{V \subset K^n : V \text{ is a } K\text{-vector space of dimension } k\}.$$

   Moreover, for a K-vector space, W, we denote by

   $$\mathbb{P}(W) := \{\text{lines in } W\}.$$

   (a) Let $V \in G_{n,k}$, show that for any basis $(e_1, ..., e_k)$ of V the element

   $$e_1 \wedge \cdots \wedge e_k \in \Lambda^k K^n$$

   is non zero, and generates a line $\lambda(V) \in \mathbb{P}(\Lambda^k K^n)$ independent of the choice of the basis.

   (b) Show that the map

   $$\begin{array}{ccc} G_{n,k} & \xrightarrow{\psi} & \mathbb{P}(\Lambda^k K^n) \\ V & \mapsto & \lambda(V), \end{array}$$

   is an injection.

   (c) Let $w \in \Lambda^k K^n$ with $w \neq 0$. Show that w is of the form

   $$v_1 \wedge \cdots \wedge v_k,$$

   for $v_i \in K^n$ if and only if the linear map

   $$\varphi_w : \begin{array}{ccc} K^n & \to & \mathbb{P}(\Lambda^{k+1} K^n) \\ v & \mapsto & w \wedge v, \end{array}$$

   has rank $n - k$.

(d) Deduce that the image of ψ is a projective subset of $\mathbb{P}(\Lambda^k K^n)$. It is called the *grassmannians of k-spaces in K^n*.

4. Let $n \geqslant 1$ be an integer

   (a) Let $0 \leqslant k \leqslant n$ and $x_1, ..., x_k \in \mathbb{P}^n$. Show that the set of lines contained in the subspace V of K^{n+1} generated by $x_1, ..., x_k$ is a projective set in $\mathbb{P}^n$. Show that it is isomorphic to $\mathbb{P}^{d-1}$, where $d = \dim(V)$. It is denoted $\mathbb{P}(x_1, ..., x_k)$.

   (b) Show that a closed projective set Y in $\mathbb{P}^n$ is isomorphic to a set of the form $\mathbb{P}(x_1, \ldots, x_k)$ for some k and some (x_i) if and only if it is the zero set of a family of homogeneous polynomials of degree $\leqslant 1$.

   (c) Let $H = \mathbb{P}^n \smallsetminus \mathbb{A}^n_{x_n}$ and $x \in \mathbb{A}^n_{x_n}$ fixed. For $y \in \mathbb{P}^n \smallsetminus \{x\}$, show that there is a unique point $\zeta \in H$ such that $\zeta \in \mathbb{P}(x, y)$.

   (d) Show that the map

$$\begin{array}{ccc} \mathbb{P}^n \smallsetminus \{x\} & \to & H \\ y & \mapsto & \zeta, \end{array}$$

   is a morphism.

5. Recall the quadric surface Q given by $xy - zw$ in $\mathbb{P}^3$ of exercise 12, sheet 1. Prove that Q is birationally equivalent to $\mathbb{P}^2$.

6. A birational map of $\mathbb{P}^2$ into itself is called a *plane Cremona transformation.* Define the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ as $[a_0 : a_1 : a_2] \mapsto [a_1 a_2 : a_0 a_2 : a_0 a_1]$.

   (a) Show that φ is birational, and its own inverse.

   (b) Find open sets $U, V \subset \mathbb{P}^2$ such that $\varphi : U \to V$ is an isomorphism.

   (c) Find the open sets where φ and φ^{-1} are defined, and describe the corresponding morphisms.

7. *Blowing-up.* We define the *Blowing-up* of $\mathbb{A}^2$ at the point 0 to be the subset $B := \{((x, y), [t : u]) \mid xu = ty\} \subset \mathbb{A}^2 \times \mathbb{P}^1$. Let $\varphi : B \to \mathbb{A}^2$ be the restriction to B of the projection onto the first component (see Figure 1). Prove that:

   (a) The map φ is birational and restricts to an isomorphism $B \smallsetminus \varphi^{-1}(0) \cong \mathbb{A}^2 \smallsetminus 0$.

   (b) We have $\varphi^{-1}(0) \cong \mathbb{P}^1$.

   (c) The points in $\varphi^{-1}(0)$ are in 1-to-1-correspondence with lines ℓ in $\mathbb{A}^2$ through the point 0. [Hint: Look at $\varphi^{-1}(\ell \smallsetminus 0)$ and its closure.]

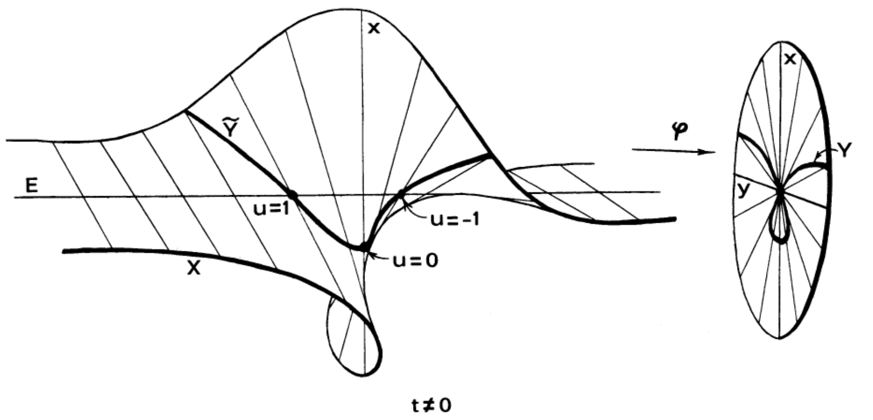

Figure 1: Blowing-up, figure taken from Hartshorne.