Algebraic Geometry

FS 2019

Exercise Sheet 3

Non Singular Varieties & Sheaves & Schemes

1. Let X be a quasi-projective irreducible variety over K.
   (a) The set of singular points of X is closed in X.
   (b) The set of singular points of X is a proper subset of X. (Consider first the case of a hypersurface).

2. Let $V = \{(x, y) : y^2 = x^3\}$. Let $\mathfrak{m} \subset \mathcal{O}(V)$ be the maximal ideal of function vanishing at 0. Show that
$$\dim_K \mathfrak{m}/\mathfrak{m}^2 = 2.$$

3. Let $f \neq 0$ be an irreducible polynomial in $K[X_1, ..., X_n]$. Assume that $f(0, ..., 0) = 0$ and that $Z(f)$ is smooth at 0. Let $m_0 \subset \mathcal{O}(Z(f))$ be the maximal ideal of function g such that $g(0) = 0$. Show that $\dim_K \mathfrak{m}_0/\mathfrak{m}_0^2 = n - 1$.

4. Let Y_1, Y_2 be closed projective sets with Y_2 irreducible. Let $d \geqslant 0$ be an integer and
$$f : Y_1 \to Y_2$$
a morphism. If $f^{-1}(y) \subset Y_1$ is irreducible of dimension d for all $y \in Y_2$,, show that Y_1 is irreducible. **Hint:** Let
$$Y_1 = \bigcup_{i=1}^{n} \tilde{Y}_i$$
the decomposition of Y_1 in irreducible component. Consider $f_i := f_{|\tilde{Y}_i}$ and apply the Theorem on the dimension of the fibers to each f_i.

5. Let X be a topological space and let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on X.
   (a) Prove that the induced map $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for every open subset $U \subset X$ if and only if the map on the stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective for every point $x \in X$.
   (b) Show that this is not true for surjectivity: Let X be the topological space $X := \mathbb{C} \setminus \{0\}$ with standard topology, let $\mathscr{F} = \mathscr{G}$ be the sheaf of nowhere-zero continuous complex-valued functions and let $\varphi : \mathscr{F} \to \mathscr{G}$ be the morphism that sends a function f to f^2. Prove that for every point $x \in X$ the induced morphism on the stalks φ_x is surjective, but on global sections $\varphi(X)$ is not surjective.