Exercise Sheet 4

Sheaves & Schemes

1. Let X be a topological space and $\mathcal{F}$ a presheaf on X. We denote by $\mathcal{F}^s$ the sheaf associated to $\mathcal{F}$. Show that the canonical map

$$\theta : \mathcal{F} \to \mathcal{F}^s,$$

induces an isomorphism

$$\theta : \mathcal{F}_x \to \mathcal{F}^s_x,$$

for all $x \in X$.

2. Let X be a topological space and $\mathcal{F}$, $\mathcal{G}$ sheaves of abelian groups on X. In the following $U \subset X$ is always an open subset of X.

   (a) If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (of abelian groups), then $\mathrm{Ker}(f) : U \to \mathrm{Ker}(f_U)$,is a sheaf (with the obvious restriction).

   (b) If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (of abelian groups), then $\mathcal{H} : U \to \mathrm{Im}(f_U)$, is not a sheaf in general. Let us denote $\mathrm{Im}(f) := \mathcal{H}^s$. Show that there is an injective morphism

$$g : \mathrm{Im}(f) \hookrightarrow \mathcal{G}.$$

3. *Inverse Image Sheaf.* Let $f : X \to Y$ be a continuous map of topological spaces. For a sheaf $\mathscr{G}$ of abelian groups on Y we define the *inverse image* sheaf $f^{-1}\mathscr{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \supset f(U)} \mathscr{G}(V)$, where U is any open set of X and the direct limit (see exercise A) is taken over all open subsets V of Y containing $f(U)$. Prove that for every sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$ and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Prove that this induces a natural bijection of sets

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y. One says that f^{-1} and f_* are adjoint functors.

4. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. Prove that if f is a homeomorphism and $f^\sharp$ is an isomorphism, then $(f, f^\sharp)$ is an isomorphism.

5. Let A be a ring and set $X := \mathrm{Spec}(A)$. Let $f \in A$ and let $U_f \subset X$ be the open complement of $V((f))$.

(a) Show that the locally ringed space $(U_f, \mathcal{O}|_{U_f})$ is isomorphic to $\mathrm{Spec}(A_f)$.

(b) For another element $g \in A$ describe the restriction map $\mathcal{O}(U_f) \to \mathcal{O}(U_{fg})$ in terms of a ring homomorphism $A_f \to A_{fg}$.

6. Consider $S_1 := \mathrm{Spec}(\mathbb{Q}[X,Y]/(XY))$ and $S_2 := \mathrm{Spec}(\mathbb{Q}[X,Y]/(X^2+Y^2))$.

   (a) Compute $\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(\mathbb{Q}), S_i)$ for $i = 1, 2$.

   (b) Deduce that S_1 and S_2 are not isomorphic schemes.

   (c) Let $S_1' := \mathrm{Spec}(\mathbb{Q}(i)[X,Y]/(XY))$ and $S_2' := \mathrm{Spec}(\mathbb{Q}(i)[X,Y]/(X^2+Y^2))$. Prove that $S_1' \cong S_2'$ as schemes.

7. Let X be a scheme. For any point $x \in X$ we define the Zariski tangent space T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k and let $k[\varepsilon]/(\varepsilon^2)$ be the *ring of dual numbers* over k. Show that to give a morphism of schemes over k of $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ to X is equivalent to giving a point $x \in X$ such that $k(x) = k$ and an element of T_x.