D-MATH | Algebraic Geometry | FS 2019
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Exercise Sheet 5

Schemes

1. Let X be a scheme and $f \in \mathcal{O}_X(X)$ a global section. Define X_f to be the subset of points $x \in X$ such that the germ f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

   (a) If $U = \operatorname{Spec}(B)$ is an open affine subscheme of X and if $\bar{f} \in B = \mathcal{O}_U(U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

   (b) Assume that X is quasi-compact. Let $A := \mathcal{O}_X(X)$ and let $a \in A$ be an element whose restriction to X_f is 0. Show that there exists an integer $n > 0$ such that $f^n a = 0$.

   (c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \mathcal{O}_{X_f}(X_f)$. Show that there exists an integer $n > 0$ such that $f^n b$ is the restriction of an element of A.

   (d) With the hypothesis of (c) conclude that $\mathcal{O}_{X_f}(X_f) \cong A_f$.

2. *A Criterion for Affineness.*

   (a) Let $f : X \to Y$ be a morphism of schemes and suppose that Y can be covered by open subsets U_i such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

   (b) A scheme X is affine if and only if there is a finite set of elements $f_1, \dots, f_r \in A := \mathcal{O}_X(X)$ such that the open subsets X_{f_i} defined in exercise 1 are affine and $f_1, \dots, f_r$ generate the unit ideal.

3. Let us recall that a *graded ring* is a ring S together with a decomposition $S = \bigoplus_{d \geqslant 0} S_d$ of S into a direct sum of abelian groups S_d, such that for any $d, e \geqslant 0$, $S_d \dot{S}_e \subset S_{de}$. An element of S_d is called a *homogeneous element of degree d*. An ideal $I \subset S$ is a *homogeneous ideal* if $I = \bigoplus_{d \geqslant 0} (I \cap S_d)$.

   (a) Let K be an algebraically closed field, and let $f_1, ..., f_m \in K[X_0, ..., X_n] = S$ be homogeneous polynomials. Re Let $I = (f_1, ...f_m) \subset S$ prove that

      i) The ideal I is an homogeneous ideal.

      ii) Show that $B := A/I$ is a graded ring.

(b) Let $B_+ = I = \bigoplus_{d \geqslant 1} B_d$. One considers

$$\mathrm{Proj}(B) := \{\mathfrak{p} \subset B \text{ homogeneous prime ideals} : B_+ \not\subseteq \mathfrak{p}\}.$$

For any homogeneous ideal, I, one defines

$$V(I) := \{\mathfrak{q} \in \mathrm{Proj}(B) : I \subset \mathfrak{q}\}.$$

Show that these sets form the closed sets of a topology on $\mathrm{Proj}(B)$.

(c) We define a sheaf of rings $\mathcal{O}$ on $\mathrm{Proj}(S)$ as follows: For each $\mathfrak{p} \in \mathrm{Proj}(S)$ we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where T is the multiplicative system of all homogeneous elments of S which are not in $\mathfrak{p}$. For any open subset $U \subset \mathrm{Proj}(S)$ we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$ we have $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and there is an open neighbourhood V of $\mathfrak{p}$ in U and homogeneous elements a, f in S of the same degree such that for all $\mathfrak{q} \in V$ we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. Prove the following:

i) For any $\mathfrak{p} \in \mathrm{Proj}(S)$, the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.

ii) For any homogeneous $f \in S_+$ let U_f^+ be the complement of $V((f))$. These open sets cover $\mathrm{Proj}(S)$ and there is an isomorphism of locally ringed spaces

$$(U_f^+, \mathcal{O}|_{U_f^+}) \cong \mathrm{Spec}(S_{(f)})$$

where $S_{(f)}$ is the subring of elements of degree zero in the localized ring S_f.

(d) Conclude that $(\mathrm{Proj}(B), \mathcal{O})$ is a scheme.