Algebraic Geometry

FS 2019

Exercise Sheet 4

Schemes

1. *Double Points.* Let k be a field and $Y \hookrightarrow \mathbb{A}^2_k$ be a closed subscheme with image containing the origin $(0,0)$ in $\mathbb{A}^2_k$ and such that $\mathcal{O}_Y(Y) \cong k[\varepsilon]/(\varepsilon^2)$. Denote by $\varphi : k[x,y] \to \mathcal{O}_Y(Y)$ the surjection defining the inclusion $Y \hookrightarrow \mathbb{A}^2$. Prove that the kernel of φ contains a non-zero element $\alpha x + \beta y$ for some $\alpha, \beta \in k$. Write $X_{\alpha,\beta} := \mathrm{Spec}(k[x,y]/\ker(\varphi))$ and show that $X_{\alpha,\beta}$ can also be characterized as the composition of the natural morphism $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)) \to \mathrm{Spec}(k[\varepsilon]) \cong \mathbb{A}^1_k$ with the inclusion of the line $\mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k$ given by $x \mapsto (\beta x, -\alpha x)$.

2. Let k be an algebraically closed field and let $Z := \mathrm{Spec}(k[X_1, \ldots, X_n]/I) \subset \mathbb{A}^n_k$ be a closed subscheme of dimension 0 supported at the origin (i.e. $\sqrt{I} = (X_1, ..., X_n)$). Furthermore, suppose that $k[X_1, \ldots, X_n]/I$ is a 3-dimensional k-vector space. Prove that Z is isomorphic to either $A := \mathrm{Spec}(k[X]/(X^3))$ or to $B := \mathrm{Spec}(k[X,Y]/(X^2, XY, Y^2))$ and that A and B are not isomorphic to each other.

3. Let $X := \mathbb{A}^2_{\mathbb{C}} \setminus \{0\} \subset \mathbb{A}^2_{\mathbb{C}}$. Prove:
   (a) The restriction map $\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}(\mathbb{A}^2_{\mathbb{C}}) \to \mathcal{O}_X(X)$ is an isomorphism.
   (b) The scheme X is not an affine scheme.

4. In the following if X is a scheme we denote by $\mathrm{sp}(X)$ the underlying topological space of X. Let S be a scheme and $\pi : X \to S$, $\rho : Y \to S$ be S-schemes. Let $\mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$ be the fiber product of sets defined by π and ρ, endowed with the topology induced by the product topology on $\mathrm{sp}(X) \times \mathrm{sp}(Y)$. We are going to study some property concerning the relation between $\mathrm{sp}(X \times Y)$ and $\mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$.
   (a) Show that we have a canonical map $f : \mathrm{sp}(X \times_S Y) \to \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$.
   (b) Show that f is surjective.
   (c) Let us consider the example $X = Y = \mathrm{Spec}\,\mathbb{C}$ and $S = \mathrm{Spec}\,\mathbb{R}$. Show that $X \times_S Y \cong \mathrm{Spec}(\mathbb{C} \oplus \mathbb{C})$ and that f is not injective.
   (d) Show that in the case of the previous Exercise, with $X = \mathrm{Spec}\,k(u)$, $Y = \mathrm{Spec}\,k(v)$ and $S = \mathrm{Spec}\,k$, the map f has infinite fibers.

(e) Let $S = \operatorname{Spec} k$ be the spectrum of an arbitrary field. By studying the example $X = Y = \mathbb{A}^1_k$, show that the image of an open subset under f is not necessarily an open subset.

5. Let $\mathcal{C}$ be a category and $X \in \mathrm{Ob}(\mathcal{C})$, one defines

$$h_X : \begin{array}{ccc} \mathcal{C} & \to & (\text{Sets}) \\ Y & \mapsto & \mathrm{Hom}(Y, X) \end{array}.$$

(a) Show that h_X is a controvariant functor.

(b) Show that any morphism $f : X_1 \to X_2$ induces a morphism of functors

$$h_f : h_{X_1} \to h_{X_2}.$$

(c) Conversely, let $\varphi : h_{X_1} \to h_{X_2}$ be a morphism of functors. There is an unique $f : X_1 \to X_2$ such that $\varphi = h_f$.

6. Let S be a scheme and consider the category $\mathcal{C} =$ (Schemes over S)

(a) If $X \to S$ is a scheme over S such that

$$h_X : T \to \mathrm{Hom}_S(T, X) = X(T)$$

is a functor to groups then X has a structure of S-group scheme.

(b) Consider $\mathbb{G}_m := \operatorname{Spec}(\mathbb{Z}[X, X^{-1}])$. Prove that

$$\mathbb{G}_m(T) = \mathcal{O}_T(T)^\times$$

for any scheme T and conclude that $\mathbb{G}_m$ is a group scheme. Moreover, describe the morphism of rings corresponding to the multiplication

$$m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m.$$