Solutions Sheet 1

Classical Varieties

Let K be an algebraically closed field. All algebraic sets below are defined over K, unless specified otherwise.

1. Describe the Zariski topology on $Z(XY) \subset \mathbb{A}^2$.

   Solution: The algebraic set $Z(XY)$ consists of the union of the two coordinate axis $Y = 0$ and $X = 0$. The proper closed subsets are given by the whole X-axis, the whole Y-axis and subsets consisting of finitely many points.

2. Assume that $\mathrm{char}(K) \neq 2, 3$. Show that the polynomial $Y^2 - X^3 - X \in K[X, Y]$ is irreducible. Describe the Zariski topology on $Z(Y^2 - X^3 - X) \subset \mathbb{A}^2$.

   Solution: We consider the polynomial as an element in the ring $(K[X])[Y]$. In the ring $K[X]$ the element X is prime and divides $X^3 + X$, but its square does not. Using Eisenstein's criterion for the irreducibility of a polynomial, we deduce that Y^2-X^3-X is irreducible. We conclude that $Z(Y^2-X^3-X) \subset \mathbb{A}^2$ is an irreducible algebraic variety. Since $K[X, Y]$ has Krull dimension 2 and (Y^2-X^3-X) is a non-zero prime ideal the coordinate ring $\mathcal{O}(Z(Y^2-X^3-X)) = K[X, Y]/(Y^2-X^3-X)$ and thus also the variety has dimension 1. Hence the only proper closed subsets are given by finitely many points.

3. Let Y be the algebraic set of $\mathbb{A}^3_K$ defined by the two polynomials $X^2 - YZ$ and $XZ - X$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

   Solution: One observes that

   $$XZ - X = 0 \Rightarrow X = 0 \text{ or } Z = 1.$$

   If $X = 0$, from the other equation one gets

   $$YZ = 0 \Rightarrow Y = 0 \text{ or } Z = 0.$$

   On the other hand, if $Z = 1$ then

   $$Y - X^2 = 0.$$

   Then one concludes that

   $$Z((X^2 - YZ, XZ - X)) = Z((X, Y)) \cap Z((X, Z)) \cap Z((Y - X^2, Z - 1))$$

To conclude the exercise, it is enough to show that $Z((X,Y)), Z((X,Z))$ and $Z((Y-X^2, Z-1))$ are irreducible varieties, i.e. that $\mathfrak{p}_1 := (X,Y), \mathfrak{p}_2 := (X,Z)$ and $\mathfrak{p}_3 := (Y-X^2, Z-1)$ are prime ideals. On the other hand, it is easy to see that

$$K[X,Y,Z]/\mathfrak{p}_i = K[T],$$

for $i = 1,2,3$, i.e. $K[X,Y,Z]/\mathfrak{p}_i$ is an integral domain for any $i = 1,2,3$. Thus $\mathfrak{p}_i$ is prime for $i = 1,2,3$.

4. Let $A \subset \mathbb{A}^n$ and $B \subset \mathbb{A}^m$ be two algebraic sets. Prove that their product set $A \times B \subset \mathbb{A}^{n+m}$ is algebraic, too.

   Solution: Let $\mathfrak{a} \subset K[X_1,\dots,X_n]$ and $\mathfrak{b} \subset K[Y_1,\dots,Y_m]$ be ideals such that $A = Z(\mathfrak{a})$ and $B = Z(\mathfrak{b})$. Let $f_1,\dots,f_r \in \mathfrak{a}$ and $g_1,\dots,g_s \in \mathfrak{b}$ be respective generators of the ideals. It is straightforward to see that $A \times B = Z(f_1,\dots,f_r,g_1,\dots,g_s)$, where we extended tacitly to the ring $K[X_1,\dots,X_n,Y_1,\dots,Y_m]$.

5. Let K be a field. An algebraic subset of $K^{n^2} = \mathcal{M}_{n\times n}(K)$ that is a subgroup of $\mathrm{GL}_n(K)$ is called a *linear algebraic group.*

   (a) Show that the following are linear algebraic group

   $$\mathrm{SL}_n(K), \quad \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a^2+b^2=1 \right\}, \quad \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}.$$

   (b) Show that if $H \subset \mathrm{SL}_n(K)$ is any subgroup then the Zariski closure is a linear algebraic group.

   (c) Let $K = \mathbb{C}$ and $n = 2$. Compute the Zariski closure of $H = \mathrm{SL}_2(\mathbb{Z})$.

   Solution:

   (a) We know that $\det A$ is a polynomial in the coefficients of $A \in \mathcal{M}_{n\times n}(K)$, thus

   $$\mathrm{SL}_n(K) = Z(\det A - 1).$$

   Let

   $$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1}, a_{2,2} & \end{pmatrix} \in \mathcal{M}_{n\times n}(K),$$

   then

   $$A \in \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a^2+b^2=1 \right\}$$

   if and only if $a_{1,1} = a_{2,2}$, $a_{1,2} = -a_{2,1}$ and $\det A = 1$. Thus

   $$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a^2+b^2=1 \right\} = Z(a_{1,1}-a_{2,2}, a_{1,2}+a_{2,1}, \det A - 1).$$

To conclude, for any $A \in \mathcal{M}_{n\times n}(K)$,

$$A \in \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\},$$

if and only if $a_{i,i} = 1$ for any $i = 1, .., n$ and $a_{i,j} = 0$ for any $1 \leqslant j < i \leqslant n$. Thus

$$\left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\} = \Big(\bigcap_{i=1}^{n} Z(a_{i,i} - 1) \Big) \cap \Big(\bigcap_{1 \leqslant j < i \leqslant n} Z(a_{i,j}) \Big).$$

(b) Let $i : \mathrm{SL}_n(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ be the inversion morphism. Since H is a subgroup we have $i : H \to H$. But i is a homeomorphism for the Zariski topology and thus maps $i : \overline{H} \to \overline{H}$. Hence $\overline{H}$ is closed under inversion. Now consider the multiplication morphism with a fixed element $x \in H$ given by $\mathrm{SL}_n(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$, $a \mapsto xa$. This is a homeomorphism, too and hence $x\overline{H} = \overline{H}$ for all $x \in H$. We conclude that for all $x \in \overline{H}$ we have $Hx \subset \overline{H}$. By the same argument as before we conclude that $\overline{H}x \subset \overline{H}$. Hence $\overline{H}$ is closed under multiplication. It follows that $\overline{H}$ is a subgroup of $\mathrm{SL}_n(\mathbb{C})$.

(c) Since $\mathbb{Z}$ is infinite and $\mathbb{A}^1_{\mathbb{C}}$ is irreducible, we conclude that the closure $\overline{\mathbb{Z}} = \mathbb{A}^1_{\mathbb{C}}$. We deduce that the closure of the subgroup $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$ and similarly the closure of $\begin{pmatrix} 1 & 0 \\ \mathbb{Z} & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ \mathbb{C} & 1 \end{pmatrix}$. From exercise 5 we know that $\overline{\mathrm{SL}_2(\mathbb{Z})}$ is a group and from the preceding we know that it contains $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \mathbb{C} & 1 \end{pmatrix}$. Notice that we have

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + ab & c + abc + a \\ b & 1 + bc \end{pmatrix}$$

Hence all matrices in $\mathrm{SL}_2(\mathbb{C})$ of the form $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with $z \neq 0$ are contained in $\overline{\mathrm{SL}_2(\mathbb{Z})}$. This is a Zariski dense subset of $\mathrm{SL}_2(\mathbb{C})$, because $\mathbb{A}^4_{\mathbb{C}} \smallsetminus Z(Z)$ is Zariski dense in $\mathbb{A}^4_{\mathbb{C}}$. We conclude that $\overline{\mathrm{SL}_2(\mathbb{Z})} = \mathrm{SL}_2(\mathbb{C})$.

6. Determine the Zariski closure for the following subsets:

(a) $\{(x, \sin(x)) \mid x \in \mathbb{C}\} \subset \mathbb{A}^2_{\mathbb{C}}$

(b) $\{(a^2 - b^2, 2ab, a^2 + b^2) \mid a, b \in \mathbb{Z}\} \subset \mathbb{A}^3_{\mathbb{C}}$.

Solution:

(a) Denote the subset by A. We claim that $\overline{A} = \mathbb{A}^2_{\mathbb{C}}$. To prove this we compute the dimension. Since A is not a finite set, we conclude that the dimension of $\overline{A}$ must be 1 or 2. Assume that it is 1. Then $\overline{A}$ is a curve. However, for all real values $-1 \leqslant y \leqslant 1$ the intersection with the irreducible curve given by $C_y := \{(x, y) \mid x \in \mathbb{C}\}$ has infinitely many points. We conclude using exercise 3 that $C_y \subset \overline{A}$ for all real $-1 \leqslant y \leqslant 1$, so $\overline{A}$ contains infinitely many different curves. But $\overline{A}$ is an algebraic set and thus must be coverable by finitely many irreducible curves which is a contradiction. We conclude that the dimension of $\overline{A}$ is 2 and since $\mathbb{A}^2_{\mathbb{C}}$ is irreducible we have proven the claim.

Aliter: Consider the associated ideal $I(\overline{A}) \subset K[X, Y]$. A polynomial $f \in I(\overline{A})$ must in particular vanish on all points $(x, \sin(x))$. Hence for any $x \in \mathbb{R}$ the polynomial $f(X, \sin(x)) \in K[X]$ has infinitely many roots $X = 2\pi\mathbb{Z} + x$. Thus $f(X, \sin(x)) = 0$. Since this holds for all $x \in \mathbb{R}$ and $\sin(\mathbb{R}) = [-1, 1]$ we conclude that $f = 0$. Hence $I(\overline{A}) = (0)$ and $\overline{A} = \mathbb{A}^2_{\mathbb{C}}$.

(b) Let A be the subset in question. These points are all zeroes of the polynomial $X^2+Y^2-Z^2$, they are Pythagorian triples. Hence $\overline{A} \subset Z(X^2+Y^2-Z^2)$. The polynomial $X^2+Y^2-Z^2$ is irreducible in $K[X, Y, Z]$, which can be seen by the Eisenstein criterion with $Y - Z$ in the ring $(K[Y, Z])[X]$. Therefore $Z(X^2 + Y^2 - Z^2)$ is irreducible of dimension 2. We only need to show that $\overline{A}$ has dimension 2. Containing infinitely many points, we know that it does not have dimension 0. Assume that it has dimension 1. For all $c \in \mathbb{Z}$, the intersection of A with the irreducible curve given by $C_c := \{(c^2 - b^2, 2cb, c^2 + b^2) \mid b \in \mathbb{C}\}$ has infinitely many points. By exercise 3 we conclude that $C_c \subset \overline{A}$ for all $c \in \mathbb{Z}$, hence $\overline{A}$ contains infinitely many different curves, which proves that it cannot have dimension 1. Thus $\overline{A}$ has dimension 2 and it follows that it is equal to $Z(X^2 + Y^2 - Z^2)$.

7. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism. This shows that not every morphism whose underlying map of topological spaces is a homeomorphism needs to be an isomorphism.

   Solution: We see that the image of φ is contained in $Z(y^2 - x^3)$. It is bijective with inverse $\varphi^{-1} : (a, b) \mapsto \frac{b}{a}$ for $a \neq 0$ and $\varphi^{-1}(0, 0) = 0$. Closed sets in $\mathbb{A}^1$ are finitely many points and they are mapped via φ to finitely many points (i.e. closed sets) in $\mathbb{A}^2$, so φ is a closed map. Hence φ and φ^{-1} are both continuous. So φ is bijective and bicontinuous. However, it is not an isomorphism. To see this let $f : \mathbb{A}^1 \to k$ be the canonical regular function $t \mapsto t$. Note that $f \circ \varphi^{-1} : (a, b) \mapsto \frac{b}{a}$ is not regular at $a = 0$, hence φ^{-1} is not a morphism.

8. Let $Y \subset \mathbb{A}^3$ be the set $Y := \{(t, t^2, t^3) \mid t \in K\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$ and prove that the coordinate ring $\mathcal{O}(Y)$ is isomorphic to a polynomial ring in one variable over K.

Solution: The ideal $I(Y)$ is equal to $(v - u^2, w - u^3) \subset K[u, v, w]$. We have the equality $Y = Z(v - u^2, w - u^3)$. Note that $K[u, v, w]/(v - u^2, w - u^3) \cong K[u, u^2, u^3] \cong K[u]$. This proves that $\mathcal{O}(Y)$ is isomorphic to a polynomial ring in one variable over K and furthermore that $(v - u^2, w - u^3)$ is a prime ideal of coheight 1, so Y is an irreducible affine variety of dimension 1.

9. (a) Show that there are not non-constant rational functions $f, g \in \mathbb{C}(X)$, such that

$$f^2 = g^3 - g.$$

(b) Show that there exist non-constant rational functions $f, g \in \mathbb{C}(X)$, such that

$$f^2 = g^3.$$

Solution:

(a) We prove the exercise when $f, g \in \mathbb{C}[X]$, the case of rational function is analogues. First of all let us recall two fact on polynomials:

i) Let $h \in \mathbb{C}[X]$ be a polynomial and let $\alpha \in \mathbb{C}$ be a root of h. We denote by $\mathrm{ord}_\alpha(h)$ *the order of vanishing of* h *at* α. If $\mathrm{ord}_\alpha(h) > 1$, then α is a root of h' of order $\mathrm{ord}_\alpha(h') = \mathrm{ord}_\alpha(h) - 1$.

ii) One has that

$$\deg h = \sum_{\alpha \in \mathbb{C}} \mathrm{ord}_\alpha(h).$$

We start proving that if $f, g \in \mathbb{C}[X]$ are such that

$$f^2 = g^3 - g,$$

then $g^2 - 1$ is a square in $\mathbb{C}[X]$. Let us start writing

$$f = a \prod_i (X - \alpha_i)^{\mathrm{ord}_\alpha(f)}$$

with $a \in \mathbb{C}$. Than for any i either $(X - \alpha_i)|g$ or $(X - \alpha_i)|g^2 - 1$. Indeed, by contradiction, if $g(\alpha_i) = g((\alpha_i))^2 - 1 = 0$ then $-1 = 0$ which is absurd. Thus we can write

$$g^2 - 1 = b \prod_{i:\ g((\alpha_i))^2 - 1 = 0} (X - \alpha_i)^{2\mathrm{ord}_\alpha(f)} = b\Big(\prod_{i:\ g((\alpha_i))^2 - 1 = 0} (X - \alpha_i)^{\mathrm{ord}_\alpha(f)} \Big)^2,$$

for some $b \in \mathbb{C}$, i.e. $g^2 - 1$ is a square. On the other hand, one has that

$$(g^2 - 1)' = g \cdot g',$$

thus

$$(g^2 - 1)' \Leftrightarrow g = 0 \text{ or } g' = 0.$$

Assume by contradiction that g is not constant. Then, using (ii) one would get

$$\deg(g^2-1)' = \sum_{\alpha\in\mathbb{C}} \operatorname{ord}_\alpha((g^2-1)') = \sum_{\alpha\in\mathbb{C}} \operatorname{ord}_\alpha(g) + \sum_{\alpha\in\mathbb{C}} \operatorname{ord}_\alpha(g') = \deg g + \sum_{\alpha\in\mathbb{C}} \operatorname{ord}_\alpha(g').$$

Since g^2-1 is a square for any $\alpha\in\mathbb{C}$ root of g^2-1 one has that $(g^2-1)'(\alpha)=0$ with $\operatorname{ord}_\alpha((g^2-1)') = \operatorname{ord}_\alpha(g^2-1)-1$ (thanks to (i)). Thus, one concludes that for any $\alpha\in\mathbb{C}$ root of g^2-1, $g'(\alpha)=0$ with $\operatorname{ord}_\alpha(g') = \operatorname{ord}_\alpha(g^2-1)-1$. Then

$$\begin{aligned}
\deg(g^2-1)' &= \deg g + \sum_{\alpha\in\mathbb{C}} \operatorname{ord}_\alpha(g') \\
&\geqslant \deg g + \sum_{\substack{\alpha\in\mathbb{C}\\ g(\alpha)^2-1=0}} \operatorname{ord}_\alpha(g') \\
&= \deg g + \sum_{\substack{\alpha\in\mathbb{C}\\ g(\alpha)^2-1=0}} (\operatorname{ord}_\alpha(g^2-1)-1) \\
&= \deg g + \sum_{\alpha\in\mathbb{C}} (\operatorname{ord}_\alpha(g^2-1) - \sum_{\substack{\alpha\in\mathbb{C}\\ g(\alpha)^2-1=0}} 1 \\
&= \deg g + \deg(g^2-1) - \sum_{\substack{\alpha\in\mathbb{C}\\ g(\alpha)^2-1=0}} 1,
\end{aligned}$$

Where in the last step we used (ii). On the other hand

$$\sum_{\substack{\alpha\in\mathbb{C}\\ g(\alpha)^2-1=0}} 1 = \text{number of distinct root of } g^2-1 \leqslant \frac{\deg(g^2-1)}{2},$$

since g^2-1 is a square. Thus, we obtain

$$\deg(g^2-1)' \geqslant \deg g + \deg(g^2-1) - \frac{\deg(g^2-1)}{2},$$

using the fact that $\deg(g^2-1) = 2\deg g$ and that $\deg((g^2-1)') = \deg(g^2-1)-1$, one obtains

$$2\deg(g) - 1 \geqslant 2\deg g,$$

which is not possible. Thus g has to be constant, and this implies that also f has to be constant. To handle the case where f, g are rational functions one argues similarly using the following facts *i)* Let $h \in \mathbb{C}(X)$ be a rational functions and let $\alpha \in \mathbb{C}$ be a zero of h. If $\text{ord}_\alpha(h) > 1$, then α is a zero of h' of order $\text{ord}_\alpha(h') = \text{ord}_\alpha(h) - 1$.

ii) Let $h \in \mathbb{C}(X)$ be a rational functions and let $\alpha \in \mathbb{C}$ be a pole of h. Then α is a pole of h' of order $\text{ord}_\alpha(h') = \text{ord}_\alpha(h) + 1$.

iii) One has that

$$\sum_{\alpha \in \mathbb{C}} \text{ord}_\alpha(h) = 0.$$

10. Let $Y \subset \mathbb{A}^3_K$ be the curve given parametrically by $x = t^3$, $y = t^4$, $z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $K[X, Y, Z]$ which cannot be generated by 2 elements. We say that Y is *not a local complete intersection.* Proceed as follows:

(a) Show that the closed subsets of Y are given by

$$\{Y\} \cup \{C \subset Y : |C| < \infty\}.$$

Conclude that Y is irreducible and that $\dim Y = 1$.

(b) Let $f \in\subset K[X, Y, Z]$, then we can write

$$f(X, Y, Z) = \sum_{n_1, n_2, n_3} c_{n_1,n_2,n_3} X^{n_1} Y^{n_2} Z^{n_3}.$$

Show that if $f \in I(Y)$, then

$$\sum_{\substack{n_1, n_2, n_3 \\ 3n_1 + 4n_2 + 5n_3 = n}} c_{n_1,n_2,n_3} = 0,$$

for any $n \geqslant 0$. Use this to show that

$$Y^2 - XZ, \quad X^3 - YZ, \quad X^2Y - Z^2 \in I(Y),$$

and deduce that Y is an affine variety.

(c) Conclude the exercise.

Solution:

(a) Let $C \subset Y$ a closed subset of Y with $|C| = \infty$. By definition, $C = Z(f_1, ..., f_n) \cap Y$ for some $f_i \in K[X, Y, Z]$. Then for any $i = 1, ..., n$, the polynomial in one variable

$$f_i(T^3, T^4, T^5)$$

vanishes for any t such that $(t^3, t^4, t^5) \in C$, i.e. $f_i(T^3, T^4, T^5)$ has infinitely many zeros since we are assuming $|C| = \infty$. Thus, for any $i = 1, ..., n$, $f_i(T^3, T^4, T^5)$ is the zero polynomial i.e. for any $i = 1, ..., n$, $f_i(t^3, t^4, t^5) = 0$

for any $t \in \mathbb{C}$. Hence, $Y \subset Z(f_1, ..., f_n)$ and this implies $Y = C$. Let us prove that Y is irreducible. Let $C_1, C_2 \subset Y$ closed subsets such that

$$Y = C_1 \cup C_2.$$

Since $|Y| = \infty$ then $|C_1| = \infty$ or $|C_2| = \infty$. Hence, $C_1 = Y$ or $C_2 = Y$ i.e. Y is irreducible. Let us prove that $\dim Y = 1$: a maximal chain of irreducible closed subset is given by

$$(0, 0, 0) \subset Y.$$

Thus, $\dim Y = 1$.

(b) Let $f \in I(Y)$, we can write f as

$$f(X, Y, Z) = \sum_{n_1, n_2, n_3} c_{n_1, n_2, n_3} X^{n_1} Y^{n_2} Z^{n_3}.$$

The polynomial in one variable

$$f(T^3, T^4, T^5) = \sum_n T^n \sum_{\substack{n_1, n_2, n_3 \\ 3n_1 + 4n_2 + 5n_3 = n}} c_{n_1, n_2, n_3}$$

has to vanish at any $t \in \mathbb{C}$ by definition of $I(Y)$. Thus $f(T^3, T^4, T^5)$ is the zero polynomial and we conclude that

$$\sum_{\substack{n_1, n_2, n_3 \\ 3n_1 + 4n_2 + 5n_3 = n}} c_{n_1, n_2, n_3} = 0, \tag{1}$$

for any $n \geqslant 0$. For the second part, it is enough to specialize equation (1) at $n = 0, ..., 10$. For example, for $n = 8$ we get

$$c_{1,0,1} - c_{0,2,0} = 0,$$

i.e. $Y^2 - XZ \in I(Y)$.

(c) Thanks to part (b) we know that $Y \subset Z(Y^2 - XZ, X^3 - YZ, X^2Y - Z^2)$. Let $(x, y, z) \in Z(Y^2 - XZ, X^3 - YZ, X^2Y - Z^2)$, we have that

$$y^2 - xz = 0, \quad x^3 - yz = 0, \quad x^2y - z^2 \in I(Y).$$

Multiplying both sides of the first equation by y we get

$$y^3 = xzy = x^4$$

thanks to the second equation. Similarly one gets that

$$z^3 = x^5, \quad z^4 = y^5.$$

Assume $x \neq 0$, and consider $t = y/x$. Then

$$t^3 = \frac{y^3}{x^3} = \frac{x^4}{x^3} = x, \quad t^4 = \frac{y^4}{x^4} = \frac{y^4}{y^3} = y, \quad t^4 = \frac{y^5}{x^5} = \frac{z^4}{z^3} = z.$$

Thus $(x, y, z) \in Y$ if $x \neq 0$. On the other hand if $x = 0$, $y = z = 0$ and $(0, 0, 0) \in Y$. Hence $Y = Z(Y^2 - XZ, X^3 - YZ, X^2Y - Z^2)$, i.e. Y is an affine variety.

(d) Let us prove that $I(Y)$ cannot be generated by two elements. By contradiction assume $I(Y) = (f, g)$ for some $f, g \in K[X, Y, Z]$. Since $(Y^2 - XZ, X^3 - YZ, X^2Y - Z^2) \subset I(Y)$, then $\deg f, \deg g \leqslant 10$. Thanks to part (*b*), we may assume without loss of generalities that

$$f = Y^2 - XZ \text{ or } f = X^3 - YZ \text{ or } f = X^2Y - Z^2$$

and that

$$g = Y^2 - XZ \text{ or } g = X^3 - YZ \text{ or } g = X^2Y - Z^2.$$

Let us discuss the case when $f = X^3 - YZ$ and $g = X^2Y - Z^2$, the others are similar. Since $I(Y) = (f, g) = (X^3 - YZ, X^2Y - Z^2)$, there would be $h, w \in K[X, Y, Z]$ such that

$$Y^2 - XZ = h \cdot (X^3 - YZ) + w \cdot (X^2Y - Z^2).$$

The equation above would be true also modulo (X, Z), i.e.:

$$Y^2 = 0 \mod (X, Z),$$

and this would imply $Y \in (X, Z)$ which is not true. Thus $Y^2 - XZ \notin (X^3 - YZ, X^2Y - Z^2)$. Similarly one proves that $X^3 - YZ \notin (Y^2 - XZ, X^2Y - Z^2)$ and $X^2Y - Z^2 \notin (Y^2 - XZ, X^3 - YZ)$. Then $\mathrm{ht}(I(Y)) = 3 - \dim Y = 2$ but $I(Y)$ cannot be generated by 2 elements.

11. *The Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_i b_j, \ldots)$ in lexicographic order, where $N = rs + r + s$. Show that ψ is well-defined and injective. It is called the *Segre embedding.* Prove that the image of ψ is a projective algebraic set in $\mathbb{P}^N$.

Solution: The map ψ is homogeneous and thus well-defined. Denote for all indices $0 \leqslant i \leqslant r$, $0 \leqslant j \leqslant s$ by x_{ij} the coordinates of a point in $\mathbb{P}^N$. The points in the image of ψ satisfy $x_{ij}x_{kl} = x_{kj}x_{il}$. Let Y be the projective algebraic set defined by these equations. Let $Q := [x_{ij}] \in Y$ be a point. Then at least one coordinate $x_{k\ell}$ is non-zero. Using that we are in projective space we have

$$Q = [x_{k\ell}x_{ij}] = [x_{i\ell}x_{kj}] = \psi([x_{0\ell} : \cdots : x_{r\ell}], [x_{k0} : \cdots : x_{ks}]).$$

Which proves that $Y = \mathrm{im}(\psi)$, hence the image of ψ is a projective algebraic set. The above also provides a left inverse to ψ which proves that ψ is injective.

12. Consider the surface Q (i.e. variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw$.

    (a) Show that Q is equal to the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

    (b) Show that Q contains two families of lines (i.e. linear varieties of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbb{P}^1$, with the property that if $L_t \neq L_u$, then $L_t \cap L_u = \varnothing$; if $M_t \neq M_u$ then $M_t \cap M_u = \varnothing$ and for all t, u we have $L_t \cap M_u =$ one point.

    (c) Show that Q contains other curves besides these lines and deduce that the Zariski topology on Q is not homeomorphic via the Segre embedding to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$.

    Solution:

    (a) The image of the Segre embedding is given by

    $$\operatorname{im}(\psi) = \left\{ [a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1] \;\middle|\; [a_0 : a_1], [b_0 : b_1] \in \mathbb{P}^1 \right\}$$

    We see that every point $[x : w : z : y]$ in the image of ψ satisfies the equation $xy - zw = 0$. Conversely suppose that a point $[x : w : z : y] \in \mathbb{P}^3$ satisfies $xy - zw = 0$. If $x \neq 0$, then the image of $([x : z], [x : w])$ under the Segre embedding is the point $[xx : xw : xz : wz] = [x : w : z : y]$. Similarly for $w \neq 0$ take $([w : y], [x : w])$, for $z \neq 0$ take $([x : z], [z : y])$ and for $y \neq 0$ take $([w : y], [z : y])$ to prove that every point $[x : w : z : y]$ with $xy - zw = 0$ is always in the image of the Segre embedding. Thus the image of the Segre embedding is equal to $Z(xy - zw) = Q$.

    (b) For any $t \in \mathbb{P}^1$ we define L_t to be $\psi(\mathbb{P}^1 \times \{t\})$ and for any $u \in \mathbb{P}^1$ we define M_u to be $\psi(\{u\} \times \mathbb{P}^1)$. We have shown in exercise 9 that ψ is injective. Thus $L_t \cap L_u = \varnothing$ and $M_t \cap M_u = \varnothing$ for $t \neq u$. Furthermore it follows that $L_t \cap M_u = \psi(u, t)$, which is one point.

    (c) The surface Q contains the curve

    $$C := Z(xy - zw, w - z) = \psi\left(\left\{ ([x : z], [x : z]) \;\middle|\; [x : z] \in \mathbb{P}^1 \right\} \right).$$

    By bijectivity of ψ, we know that $C \cap L_t$ and $C \cap M_t$ are both one point for all $t \in \mathbb{P}^1$, hence we conclude that C is a different curve. The curve C is closed in $\mathbb{P}^3$, but the set $\{([x : z], [x : z]) \mid [x : z] \in \mathbb{P}^1\}$ is not closed in the product topology of $\mathbb{P}^1 \times \mathbb{P}^1$, because it is the diagonal of a non-Hausdorff space.

13. Let $n, d > 0$ be integers. We denote by $M_0, \ldots, M_N$ all monomials of degree d in the $n + 1$ variables $x_0, \ldots, x_n$, where $N := \binom{n+d}{n} - 1$. We define the d-uple embedding as the map $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ sending the point $a = (a_0, \ldots, a_n)$

to the point $(M_0(a), \dots, M_N(a))$. Show that the d-uple embedding of $\mathbb{P}^n$ is an isomorphism onto its image.

[Hint: Look at the inverse map.]

Solution: For $0 \leqslant i, j \leqslant n$ denote by M_{ij} the monomial $x_i^{d-1}x_j$ and denote for a point $[b_0 : \cdots : b_N]$ in the image of ρ_d by b_{ij} the coordinate corresponding to the monomial M_{ij}. Note that at least one b_{ii} is non-zero. On the chart $b_{ii} \neq 0$, define the map $\varphi_i : [b_0 : \cdots : b_N] \mapsto [b_{i0} : \cdots : b_{in}]$. These maps φ_i glue to an inverse of ρ_d on the whole image. Since they are defined only by projecting to certain coordinates, they are regular and thus glue to a morphism. Hence ρ_d is an isomorphism onto its image.