Solutions Sheet 2

Classical Varietes, Rational Maps, Blowups, Spectrum

Let K be an algebraically closed field. All algebraic sets and varieties below are defined over K, unless specified otherwise.

1. Consider the set $M := \mathrm{Mat}_{m,n}(K)$ of $m \times n$-matrices. It can be identified with the affine algebraic variety $\mathbb{A}^{nm}$. Determine if S is open/closed/dense in M:

   (a) $S := \{A \in M \mid A^t A \text{ has an eigenvalue } 1\}$

   (b) $S := \{A \in M \mid \mathrm{rank}(A) = \min\{m, n\}\}$

   (c) for $m = n$, $S := \{A \in M \mid A \text{ is diagonalisable}\}$

   Solution:

   (a) We define the map $\varphi : M \to \mathbb{A}^1$ to be $A \mapsto \det(A^t A - \mathrm{id})$. For a matrix $A \in M$ the matrix $A^t A$ has an eigenvalue 1 if and only if $\varphi(A) = 0$. The map φ is a polynomial in the coefficients of A and hence a morphism of algebraic varieties. We conclude that $S = \varphi^{-1}(0)$ is a closed subset of M.

   (b) We define $d := \min\{m, n\}$ and $N := \binom{\max\{m,n\}}{d}$. We define $\varphi : M \to \mathbb{A}^N$ as the map taking an $m \times n$-matrix A to all of its $d \times d$-minors. Then A has full $\mathrm{rank}(A) = d$ if and only if $\varphi(A) \neq (0, \ldots, 0)$. Since minors are polynomial expressions in the coefficients of A, we conclude that φ is a morphism of algebraic varieties. Hence $S = M \smallsetminus \varphi^{-1}(0)$ is an open subset of M.

   (c) It is neither: We define the map $\varphi : M \to \mathbb{A}^1$ as the map taking a matrix $A \in M$ to the discriminant of its characteristic polynomial. Since the discriminant is a polynomial expression in the coefficients of the characteristic polynomial, and these coefficients are polynomial expressions in the coefficients of A, we conclude that φ is a morphism of algebraic varieties. Furthermore, every matrix A with $\varphi(A) \neq 0$ is diagonalisable. Thus S contains an open set $S \supset \varphi^{-1}(0)$ and since M is irreducible, we conclude that S is dense in M. Since not all matrices are diagonalisable, we conclude that S is not closed in M. To see that S is neither open, for all $\alpha \in K$ define the matrices A_α as everywhere zero except for the upper left corner, where we have $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. Define the set R to be the set of matrices A_α for $\alpha \neq 0$. Then every element of R is not diagonalisable and thus in the complement of S. However, the matrix A_0 is contained in the Zariski closure of R and is diagonalisable. Hence the complement of S is not closed, which implies that S is not open.

2. Show that a commutative K-algebra, A, is of the form $\mathcal{O}(Y)$ for some algebraic set Y if and only if it is finitely generated and it does not contains non-zero nilpotent elements.

   Solution: Let A be a finitely generated commutative K-algebra which does not contain non-zero nilpotent. Since A is finitely generated, there exists a set of generators $x_1, ..., x_n$ over K. Let $K[X_1, ..., X_n]$ be a the polynomial ring in n variables, then the ring morphism

$$\psi : \begin{array}{ccc} K[X_1, ..., X_n] & \to & A \\ X_i & \mapsto & x_i, \end{array}$$

   is surjective, i.e. $A = K[X_1, ..., X_n]/\operatorname{Ker}(\psi)$. Moreover $\operatorname{Ker}(\psi)$ is a radical ideal, since A does not contains non-zero nilpotent elements. Thus, $A = O(Y)$ where $Y = Z(\operatorname{Ker}(\psi))$. On the other hand, assume that $A = O(Y)$ for some algebraic set Y. We may assume $Y \subset \mathbb{A}^n_K$ for some $n \geqslant 0$. Then there exists $I \subset K[X_1, ..., X_n]$ ideal such that $Y = Z(I)$. Then by definition $A = O(Y) = K[X_1, ..., X_n]/\sqrt{I}$, i.e. A is a finitely generated commutative K-algebra which does not contain non-zero nilpotent.

3. Let $n \geqslant 1$ be an integer and $1 \leqslant k \leqslant n$. Let

$$G_{n,k} := \{V \subset K^n : V \text{ is a } K\text{-vector space of dimension } k\}.$$

   Moreover, for a K-vector space, W, we denote by

$$\mathbb{P}(W) := \{\text{lines in } W\}.$$

   (a) Let $V \in G_{n,k}$, show that for any basis $(e_1, ..., e_k)$ of V the element

$$e_1 \wedge \cdots \wedge e_k \in \Lambda^k K^n$$

   is non zero, and generates a line $\lambda(V) \in \mathbb{P}(\Lambda^k K^n)$ independent of the choice of the basis.

   (b) Show that the map

$$\begin{array}{ccc} G_{n,k} & \xrightarrow{\psi} & \mathbb{P}(\Lambda^k K^n) \\ V & \mapsto & \lambda(V), \end{array}$$

   is an injection.

   (c) Let $w \in \Lambda^k K^n$ with $w \neq 0$. Show that w is of the form

$$v_1 \wedge \cdots \wedge v_k,$$

   for $v_i \in K^n$ if and only if the linear map

$$\varphi_w : \begin{array}{ccc} K^n & \to & \mathbb{P}(\Lambda^{k+1} K^n) \\ v & \mapsto & w \wedge v, \end{array}$$

   has rank $n - k$.

(d) Deduce that the image of ψ is a projective subset of $\mathbb{P}(\Lambda^k K^n)$. It is called the *grassmannians of k-spaces in K^n*.

Solution: First, we recall some notion about exterior algebras:

i) for any $v_1, ..., v_k \in K^n$ one has that

$$v_1 \wedge \cdots \wedge v_k \neq 0$$

if and only if $v_1, ..., v_k$ are linearly independent.

ii) for any $v_1, ..., v_k \in V$ and any $a_1, ..., a_k \in V$

$$a_1 v_1 \wedge \cdots \wedge a_k v_k = \Big(\prod_{i}^{k} a_i\Big) v_1 \wedge \cdots \wedge v_k$$

iii) for any $v_1, ..., v_k \in V$ and any $\sigma \in \Omega_k$ (Ω_k is the permutation group of k elements), we have

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \mathrm{sng}(\sigma) v_1 \wedge \cdots \wedge v_k$$

iv) A basis for $\Lambda^k K^n$ is given by

$$\mathcal{E}_k := \{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leqslant i_1 < \cdots < i_k \leqslant n\},$$

where $\{e_1, ..., e_n\}$ is the standard basis of K^n.

(a) Let $\{e_1, ..., e_k\}$ and $\{f_1, ..., f_k\}$ be two basis for V. Then for any $i = 1, ..., k$ we have that

$$f_i = \sum_{j=1}^{k} a_{i,j} e_j,$$

for some $a_{i,j} \in K$. Then

$$
\begin{aligned}
f_1 \wedge \cdots \wedge f_k &= \Big(\sum_{j_1=1}^{k} a_{1,j_1} e_{j_1}\Big) \wedge \cdots \wedge \Big(\sum_{j_k=1}^{k} a_{k,j_k} e_{j_k}\Big) \\
&= \sum_{j_1=1}^{k} \cdots \sum_{j_k=1}^{k} a_{1,j_1} e_{j_1} \wedge \cdots \wedge a_{1,j_k} e_{j_k} \\
&\overset{(i)}{=} \sum_{\substack{j_1=1 \\ j_s \neq j_t \text{ if } s \neq t}}^{k} \cdots \sum_{j_k=1}^{k} a_{1,j_1} e_{j_1} \wedge \cdots \wedge a_{1,j_k} e_{j_k} \\
&= \sum_{\sigma \in \Omega_k} a_{1,\sigma(1)} e_{\sigma(1)} \wedge \cdots \wedge a_{1,\sigma(k)} e_{\sigma(k)} \\
&\overset{(ii)}{=} \sum_{\sigma \in \Omega_k} \Big(\prod_{i=1}^{k} a_{i,\sigma(i)}\Big) e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)} \\
&\overset{(iii)}{=} \Big(\sum_{\sigma \in \Omega_k} \operatorname{sng}(\sigma) \prod_{i=1}^{k} a_{i,\sigma(i)}\Big) e_1 \wedge \cdots \wedge e_k \\
&= (\det A) e_1 \wedge \cdots \wedge e_k,
\end{aligned}
$$

where $A = (a_{i,j})_{1 \leqslant i,j \leqslant k}$. Thus $e_1 \wedge \cdots \wedge e_k$ and $f_1 \wedge \cdots \wedge f_k$ generate the same line in $\mathbb{P}(\Lambda^k K^n)$.

(b) Let $V, W \subset K^n$ be two subvector spaces of K^n of dimension k, and assume $V \neq W$. Let $\{e_1, ..., e_k\}$ be a basis for V and $\{f_1, ..., f_k\}$ be a basis for W. Since $V \neq W$, there exists an element of the basis of W, f_i, such that $f_i \notin V$. Thus, $e_1, ..., e_k, f_i$ are linearly independent, i.e.

$$e_1 \wedge \cdots \wedge e_k \wedge f_i \neq 0.$$

By contradiction, assume $\lambda(V) = \lambda(W)$. Then one would have

$$a(e_1 \wedge \cdots \wedge e_k) = f_1 \wedge \cdots \wedge f_k,$$

for some $a \in K^\times$. This would imply that

$$0 = f_1 \wedge \cdots \wedge f_k \wedge f_i = a(e_1 \wedge \cdots \wedge e_k \wedge f_i) \neq 0,$$

and this is absurd. Then $\lambda(V) \neq \lambda(W)$ if $V \neq W$, i.e. the map λ is injective.

(c) Let us first assume that $w = v_1 \wedge \cdots \wedge v_k$ for some $v_1, ..., v_k \in K^n$. Then there exist $u_{k+1}, ..., u_n \in K^n$ such that $v_1, ..., v_k, u_{k+1}, ..., u_n$ is a basis for K^n. Then we have that $\varphi_w(v_i) = v_1 \wedge \cdots v_i \cdots \wedge v_k \wedge v_i = 0$ for any $i = 1, ..., k$. On the other hand

$$\{\varphi_w(u_j) : j = k+1, ..., n\} = \{v_1 \wedge \cdots \wedge v_k \wedge u_j : j = k+1, ..., n\} \subset \mathcal{E}_k$$

is a subset of $n-k$ elements of the basis for $\Lambda^k K^n$. Thus $\mathrm{rank}(\varphi_w) = n-k$. Assume that $w \in \Lambda^k K^n$ is such that φ_w has rank $n-k$. Then, there exists a basis $v_1, ..., v_n$ such that $\varphi_w(v_i) = 0$ if $1 \leqslant i \leqslant k$ and $\varphi_w(v_i) \neq 0$ if $k+1 \leqslant i \leqslant n$. Then we have

$$\begin{aligned} w &= \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} a_{i_1,\ldots,i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} \\ &= a_{1,\ldots,k} v_1 \wedge \cdots \wedge v_k + \sum_{\substack{1 \leqslant i_1 < \cdots < i_k \leqslant n \\ i_k > k}} a_{i_1,\ldots,i_k} v_{i_1} \wedge \cdots \wedge v_{i_k}. \end{aligned}$$

Let $1 \leqslant s \leqslant k$. By hypothesis, one knows that $w \wedge v_s = 0$. Hence, one has that

$$\begin{aligned} 0 = w \wedge v_s &= a_{1,\ldots,k} v_1 \wedge \cdots \wedge v_k \wedge v_s + \sum_{\substack{1 \leqslant i_1 < \cdots < i_k \leqslant n \\ i_k > k}} a_{i_1,\ldots,i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v_s \\ &= \sum_{\substack{1 \leqslant i_1 < \cdots < i_k \leqslant n \\ i_j \neq s \text{ for any } i_j}} a_{i_1,\ldots,i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v_s. \end{aligned} \tag{1}$$

On the other hand.

$$\{v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v_s : 1 \leqslant i_1 < \cdots < i_k \leqslant n,\ i_j \neq s \text{ for any } i_j\} \subset \mathcal{E}_{k+1}$$

Thus equation (**??**) implies

$$a_{a_{i_1,\ldots,i_k}} = 0$$

for any $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that $i_j \neq s$ for any i_j. Since this holds for any $1 \leqslant s \leqslant k$ we conclude that

$$w = a_{1,\ldots,k} v_1 \wedge \cdots \wedge v_k,$$

as we wanted.

(d) The map

$$\varphi : \begin{array}{ccc} \Lambda^k K^n & \to & \mathrm{Hom}(V, \Lambda^{k+1} K^n) \\ w & \mapsto & \varphi_w, \end{array}$$

is linear, that is, the entries of the matrix $\varphi_w \in \mathrm{Hom}(V, \Lambda^{k+1} K^n)$ are homogeneous coordinates on $\mathbb{P}(\Lambda^k K^n)$; we say that $G_{n,k} \subset \mathbb{P}(\Lambda^k K^n)$ is the subvariety defined by the vanishing of the $(n-k+1) \times (n-k+1)$ minors of this matrix.

4. Let $n \geqslant 1$ be an integer

   (a) Let $0 \leqslant k \leqslant n$ and $x_1, ..., x_k \in \mathbb{P}^n$. Show that the set of lines contained in the subspace V of K^{n+1} generated by $x_1, ..., x_k$ is a projective set in $\mathbb{P}^n$. Show that it is isomorphic to $\mathbb{P}^{d-1}$, where $d = \dim(V)$. It is denoted $\mathbb{P}(x_1, ..., x_k)$.

(b) Show that a closed projective set Y in $\mathbb{P}^n$ is isomorphic to a set of the form $\mathbb{P}(x_1, \dots, x_k)$ for some k and some (x_i) if and only if it is the zero set of a family of homogeneous polynomials of degree $\leqslant 1$.

(c) Let $H = \mathbb{P}^n \smallsetminus \mathbb{A}^n_{x_n}$ and $x \in \mathbb{A}^n_{x_n}$ fixed. For $y \in \mathbb{P}^n \smallsetminus \{x\}$, show that there is a unique point $\zeta \in H$ such that $\zeta \in \mathbb{P}(x, y)$.

(d) Show that the map

$$\begin{array}{ccc} \mathbb{P}^n \smallsetminus \{x\} & \to & H \\ y & \mapsto & \zeta, \end{array}$$

is a morphism.

Solution:

(a) Let $V \subset K^n$ be the subspace of dimension d generate by $x_1, ..., x_k$. Without loss of generalities, we may assume that $x_1, ..., x_d$ generates V. Then a vector v is contained in V if and only if all the $(d+1) \times (d+1)$ minors of the matrix

$$A = \begin{pmatrix} v \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

have determinant vanishing. These determinants are homogeneus linear equations in the variables $v = (v_1, ..., v_n)$, thus V is a projective varieties. For the second part, let us consider $u_1, ..., u_d$ a basis for V. Recall that we can see

$$\mathbb{P}^n = G_{n,1} = \{\langle v \rangle : v \in K^{n+1} \smallsetminus \{\mathbf{0}\}\}.$$

Then we define

$$\varphi : \begin{array}{ccc} \mathbb{P}^d & \to & \mathbb{P}^n \\ [a_1 : ... : a_d] & \mapsto & \langle a_1 u_1 + \cdots + a_d u_d \rangle. \end{array}$$

It is easy to check that this map is an injective morphism and that $\varphi(\mathbb{P}^d) = \mathbb{P}(x_1, ..., x_k)$.

(b) In part (a) we have shown that, if $Y = \mathbb{P}(x_1, ..., x_k)$ for some k and some (x_i), then Y is the zero set of a family of homogeneous polynomials of degree $\leqslant 1$. On the other hand, let Y be the zero set of homogeneous polynomials of degree $\leqslant 1$, $f_1, ..., f_n$. Let us consider $I = (f_1, ..f_n)$. Then $Z(I) \subset K^{n+1}$ is a subvectorspace of K^{n+1}, because $f_1, ..., f_n$ are linear polynomials. Then $Y = \mathbb{P}(x_1, ..., x_{\dim(Z(I))})$ where $\{x_1, ..., x_{\dim(Z(I))}\}$ is a basis for $Z(I)$.

(c) Let $x \in \mathbb{A}^n_{x_n}$ and $y \in \mathbb{P}^n \smallsetminus \{x\}$. Then $\mathbb{P}(x, y)$ is the projective line passing through x and y. We can parametrize this line as

$$\mu \cdot x + \lambda y.$$

Then we have that $\mu \cdot x + \lambda y \in H$ if and only if

$$\mu x_n + \lambda y_n = 0.$$

Since $x \in \mathbb{A}^n_{x_n}$, the above equation became

$$\mu = -\lambda y_n.$$

Thus, we have two possibilities: either $\zeta = y \in H \cap \mathbb{P}(x,y)$ or $\zeta := y - x \in H \cap \mathbb{P}(x,y)$. Moreover, in both cases this point is unique.

(d) Let us consider the map

$$\psi : \begin{array}{ccc} \mathbb{P}^n \smallsetminus \{x\} & \to & H \\ y & \mapsto & \zeta, \end{array}$$

Let $Z(f) \subset H$ be a closed subset where f is an homogeneous polynomial. Since $H = \mathbb{P}^n \smallsetminus \mathbb{A}^n_{x_n}$, then $f \in K[X_1, ..., X_{n-1}]$. Our goal is to show that $\psi^{-1}(Z(f))$ is closed in $\mathbb{P}^n \smallsetminus \{x\}$. Let us first consider $\psi^{-1}(Z(f)) \cap \mathbb{A}^n_{x_n}$. Thanks to part (c), $y \in \psi^{-1}(Z(f)) \cap \mathbb{A}^n_{x_n}$ if and only if $y - x \in Z(f)$, i.e. if and only if y is solution of the polynomial

$$g := f(X_1 - x_1, ..., X_{n-1} - x_{n-1}).$$

Let G be the polynomial

$$G = f(X_1 - x_1 X_n, ..., X_{n-1} - x_{n-1} X_n).$$

Then we have that

$$Z(G) \cap H = Z(f),$$

because

$$G(X_1, ..., X_{n-1}, 0) = f(X_1, ..., X_{n-1}).$$

On the other hand, $Z(G) \cap \mathbb{A}^n_{x_n} = \psi^{-1}(Z(f)) \cap \mathbb{A}^n_{x_n}$, since

$$G(X_1, ..., X_{n-1}, 1) = f(X_1 - x_1, ..., X_{n-1} - x_{n-1}).$$

Thus, $\psi^{-1}(Z(f)) = Z(G) \cap \mathbb{P}^n \smallsetminus \{x\}$, i.e. $\psi^{-1}(Z(f))$ is closed in $Z(G) \cap \mathbb{P}^n \smallsetminus \{x\}$. Hence, ψ is a morphism since

$$\{V(f) : f \in K[X_1, ..., X_{n-1}] \text{ homogeneous polynomial}\}$$

is a base for the Zarisky topology of H.

5. Recall the quadric surface Q given by $xy - zw$ in $\mathbb{P}^3$ of exercise 12, sheet 1. Prove that Q is birationally equivalent to $\mathbb{P}^2$.

Solution: Let $U \subset Q$ be the open set defined as the complement $Q \setminus \{[0:0:0:1]\}$. We define the map $f : U \to \mathbb{P}^2$ as $[x:y:z:w] \mapsto [x:y:z]$. This is well-defined on U and thus defines a rational map from Q to $\mathbb{P}^2$. Let $V \subset \mathbb{P}^2$ be the open set where the third coordinate does not vanish. Define the map $g : V \to Q$ as $[x_0 : x_1 : x_2] \mapsto [x_0x_2 : x_1x_2 : x_2^2 : x_0x_1]$. It is well-defined on V and thus defines a rational map from $\mathbb{P}^2$ to Q. We note that the compositions $f \circ g$ and $g \circ f$ are the identity on the respective open sets where they are defined. Hence Q and $\mathbb{P}^2$ are birationally equivalent.

6. A birational map of $\mathbb{P}^2$ into itself is called a *plane Cremona transformation*. Define the rational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ as $[a_0 : a_1 : a_2] \mapsto [a_1a_2 : a_0a_2 : a_0a_1]$.

   (a) Show that φ is birational, and its own inverse.

   (b) Find open sets $U, V \subset \mathbb{P}^2$ such that $\varphi : U \to V$ is an isomorphism.

   (c) Find the open sets where φ and φ^{-1} are defined, and describe the corresponding morphisms.

   Solution:

   (a) Define the open set $V \subset \mathbb{P}^2$ to be the complement in $\mathbb{P}^2$ of the three points $[1:0:0], [0:1:0], [0:0:1]$. Then φ is regular on V, hence a rational map on $\mathbb{P}^2$. Let $[a_0 : a_1 : a_2]$ be a point in the preimage $\varphi^{-1}(V) \subset V$. Then

   $$\varphi^2([a_0 : a_1 : a_2]) = [a_0^2a_1a_2 : a_0a_1^2a_2 : a_0a_1a_2^2] = [a_0 : a_1 : a_2].$$

   So φ is birational and its own inverse.

   (b) Define the subset $U := \mathbb{P}^2 \setminus V(xyz)$ as the set of all points in $\mathbb{P}^2$ where no coordinate is zero. Then $\varphi(U) \subset U$ and by the above calculation, φ is an isomorphism from U to U. On U, the isomorphism φ is given by $[a_0 : a_1 : a_2] \mapsto [\frac{1}{a_0} : \frac{1}{a_1} : \frac{1}{a_2}]$.

   (c) The maps φ and $\varphi^{-1} = \varphi$ are both maximally defined on the open set V given in the solution of (a).

7. *Blowing-up.* We define the *Blowing-up* of $\mathbb{A}^2$ at the point 0 to be the subset $B := \{((x,y),[t:u]) \mid xu = ty\} \subset \mathbb{A}^2 \times \mathbb{P}^1$. Let $\varphi : B \to \mathbb{A}^2$ be the restriction to B of the projection onto the first component (see Figure 1). Prove that:

   (a) The map φ is birational and restricts to an isomorphism $B \setminus \varphi^{-1}(0) \cong \mathbb{A}^2 \setminus 0$.

   (b) We have $\varphi^{-1}(0) \cong \mathbb{P}^1$.

   (c) The points in $\varphi^{-1}(0)$ are in 1-to-1-correspondence with lines ℓ in $\mathbb{A}^2$ through the point 0. [Hint: Look at $\varphi^{-1}(\ell \setminus 0)$ and its closure.]

   Solution:

(a) The projection is a morphism and thus in particular a rational map. Define the open set $U := \mathbb{A}^2 \setminus 0$ and the morphism $\psi : U \to B$ given by $(x, y) \mapsto ((x, y), [x : y])$. The map ψ defines a rational map from $\mathbb{A}^2$ to B which is clearly the inverse of φ. Therefore φ is birational. Looking at the definition of ψ, we see that $\psi(U) \subset B \setminus \varphi^{-1}(0)$. Thus φ restricts to an isomorphism with inverse ψ.

(b) Since φ is projection onto the first component and $(0, [t : u]) \in B$ for all $[t : u] \in \mathbb{P}^1$, it follows that $\varphi^{-1}(0) = \{0\} \times \mathbb{P}^1 \cong \mathbb{P}^1$.

(c) Let $\ell \subset \mathbb{A}^2$ be a line through the point 0 given by $\ell = \{(az, bz) \mid z \in \mathbb{A}^1\}$ for two parameters $a, b \in K$ which are not both zero, i.e. $[a : b] \in \mathbb{P}^1$. The inverse image $\varphi^{-1}(\ell \setminus 0) = \psi(\ell \setminus 0)$ is given by $\{((az, bz), [az : bz]) \mid z \in \mathbb{A}^1 \setminus 0\}$. But in this set we have $[az : bz] = [a : b]$ which is also defined for $z = 0$. Hence the closure of $\varphi^{-1}(\ell \setminus 0)$ contains the point $((0, 0), [a : b])$. So we have a 1-to-1-correspondence given by sending ℓ to $((0, 0), [a : b])$.

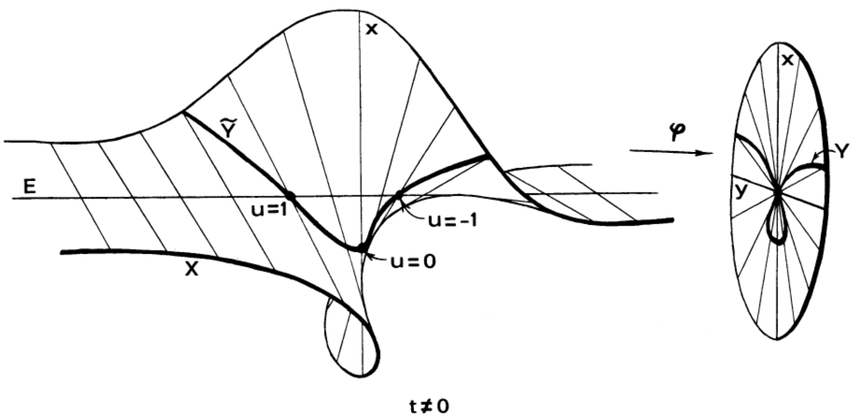

Figure 1: Blowing-up, figure taken from Hartshorne.