Solutions Sheet 3

Non Singular Varieties & Sheaves & Schemes

1. Let X be a quasi-projective irreducible variety over K.

   (a) The set of singular points of X is closed in X.

   (b) The set of singular points of X is a proper subset of X. (Consider first the case of a hypersurface).

   Solution: This is Theorem 5.3 in Hartshorne's "Algebraic Geometry".

2. Let $V = \{(x, y) : y^2 = x^3\}$. Let $\mathfrak{m} \subset \mathcal{O}(V)$ be the maximal ideal of function vanishing at 0. Show that
$$\dim_K \mathfrak{m}/\mathfrak{m}^2 = 2.$$

   Solution: Let us denote by $\overline{X}, \overline{Y}$ the image of $X, Y \in \mathcal{O}(V)$ in $\mathfrak{m}/\mathfrak{m}^2$. By contradiction, assume that $\overline{X}, \overline{Y}$ are linearly dependent in $\mathfrak{m}/\mathfrak{m}^2$. Then there exists a $f \in \mathfrak{m}^2$ and $a \in K$ such that
$$Y = aX + f.$$
   On the other hand, $\mathfrak{m}/\mathfrak{m}^2 = (X^2, YX, Y^2) = (X^2, YX)$ since $X^3 = Y^2$. Thus $f(0, y) = 0$ for any $y \in K$. Evaluating both sides of the equation above at the point $(0, 1)$, one gets $1 = 0$ which is absurd.

3. Let $f \neq 0$ be an irreducible polynomial in $K[X_1, ..., X_n]$. Assume that $f(0, ..., 0) = 0$ and that $Z(f)$ is smooth at 0. Let $m_0 \subset \mathcal{O}(Z(f))$ be the maximal ideal of function g such that $g(0) = 0$. Show that $\dim_K \mathfrak{m}_0/\mathfrak{m}_0^2 = n - 1$.

   Solution: Let us denote by $\overline{X}_1, ..., \overline{X}_n$ the image of $X_1, ..., X_n$ in $\mathfrak{m}_0/\mathfrak{m}_0^2$. Clearly $\overline{X}_1, ..., \overline{X}_n$ generates $\mathfrak{m}_0/\mathfrak{m}_0^2$ since $\mathfrak{m}_0 = (X_1, ..., X_n)$. We start proving that $\dim_K \mathfrak{m}_0/\mathfrak{m}_0^2 \leqslant n - 1$. One writes f as
$$\begin{aligned} f &= \sum_{d=1}^{D} \sum_{\substack{h \in K[X_1,...,X_n] \\ h \text{ homogeneous} \\ \deg h = d}} h \\ &= b + \sum_{j=1}^{n} a_j X_j + \sum_{d=2}^{D} \sum_{\substack{h \in K[X_1,...,X_n] \\ h \text{ homogeneous} \\ \deg h = d}} h, \end{aligned}$$

where D is the total degree of f and $b, a_1, ..., a_n \in K$. First of all, $b = 0$ since $(0, ..., 0) \in Z(f)$ by hypothesis. For any $j = 1, ..., n$ one has that

$$\frac{\partial f}{\partial X_j} = a_j + \sum_{d=2}^{D} \sum_{\substack{h \in K[X_1,...,X_n] \\ h \text{ homogeneous} \\ \deg h = d}} \frac{\partial h}{\partial X_j}.$$

Now we distinguish two cases

(a) $\deg_{X_j} h = 0$, then $\frac{\partial h}{\partial X_j} = 0$,

(b) $\deg_{X_j} h > 0$, then $\deg \frac{\partial h}{\partial X_j} \geqslant 1$ since $\deg h \geqslant 2$.

In both cases, one concludes that $\frac{\partial h}{\partial X_j}(0, ..., 0) = 0$, for any h appearing in the decomposition of f with $\deg h \geqslant 2$. Then

$$\begin{cases} \frac{\partial f}{\partial X_1}(0, ..., 0) = a_1 \\ \vdots \\ \frac{\partial f}{\partial X_j}(0, ..., 0) = a_n. \end{cases}$$

Since $(0, ..., 0)$ is a non singular point in $Z(f)$, there exists $i \in \{1, ..., n\}$ such that $a_i \neq 0$. Thus

$$X_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} a_j a_i^{-1} X_j - a_i^{-1} \sum_{d=2}^{D} \sum_{\substack{h \in K[X_1,...,X_n] \\ h \text{ homogeneous} \\ \deg h = d}} h,$$

in $O(Z(f))$ which implies

$$\overline{X}_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} a_j a_i^{-1} \overline{X}_j,$$

in $\mathfrak{m}_0/\mathfrak{m}_0^2$. Hence, $\mathfrak{m}_0/\mathfrak{m}_0^2 = (\overline{X}_1, ..., \overline{X}_{i-1}, \overline{X}_{i+1}, ..., \overline{X}_n)$, i.e. $\dim_K \mathfrak{m}_0/\mathfrak{m}_0^2 \leqslant n - 1$. To prove that $\dim_K \mathfrak{m}_0/\mathfrak{m}_0^2 \geqslant n - 1$ one applies Corollary 11.15 in Atiyah and Macdonald's "Introduction of Commutative Algebra".

4. Let Y_1, Y_2 be closed projective sets with Y_2 irreducible. Let $d \geqslant 0$ be an integer and

$$f : Y_1 \to Y_2$$

a morphism. If $f^{-1}(y) \subset Y_1$ is irreducible of dimension d for all $y \in Y_2$,, show that Y_1 is irreducible. **Hint:** Let

$$Y_1 = \bigcup_{i=1}^{n} \tilde{Y}_i$$

the decomposition of Y_1 in irreducible component. Consider $f_i := f_{|\tilde{Y}_i}$ and apply the Theorem on the dimension of the fibers to each f_i.

Solution: This is Theorem 11.14 In Harris' "Algebraic Geometry- A First Course".

5. Let X be a topological space and let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on X.

   (a) Prove that the induced map $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for every open subset $U \subset X$ if and only if the map on the stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective for every point $x \in X$.

   (b) Show that this is not true for surjectivity: Let X be the topological space $X := \mathbb{C} \smallsetminus \{0\}$ with standard topology, let $\mathscr{F} = \mathscr{G}$ be the sheaf of nowhere-zero continuous complex-valued functions and let $\varphi : \mathscr{F} \to \mathscr{G}$ be the morphism that sends a function f to f^2. Prove that for every point $x \in X$ the induced morphism on the stalks φ_x is surjective, but on global sections $\varphi(X)$ is not surjective.

   Solution:

   (a) Suppose $\varphi(U)$ is injective for every open subset $U \subset X$. Let $x \in X$ be a point. An element in the stalk $\mathscr{F}_x$ is an equivalence class of pairs (V, s_V) for all open neighbourhoods $V \subset X$ of x and $s_V \in \mathscr{F}(V)$. Assume that $\varphi_x([V, s_V]) = 0$. Then there exists an open neighbourhood $W \subset X$ of x such that $\varphi_x([V, s_V]) = [W, 0]$. But then $\varphi(W)(s_W) = 0$ and so $s_W = 0$. We conclude that $[V, s_V] = 0$.

   Conversely suppose φ_x is injective for all points $x \in X$. Let $U \subset X$ be an open subset and let $s \in \mathscr{F}(U)$ be a section. Assume that $\varphi(U)(s) = 0$. Then for any point $x \in U$ we have $\varphi_x([U, s]) = 0$, hence $s|_V = 0$ for some open neighbourhood $V \subset U$ of x by injectivity of φ_x. Doing this for all points $x \in U$ we conclude that s is zero on an open cover of U. Since $\mathscr{F}$ is a sheaf we conclude that $s = 0$.

   (b) On global sections this morphism is not surjective, because the function $z \mapsto z$ does not have a preimage. To see this, note that any root function $z \mapsto \sqrt{z}$ is not continuous on all of $\mathbb{C} \smallsetminus \{0\}$. However, there is a preimage of every element in the stalk. To see this let $[U, f]$ be an element of $\mathscr{F}_x$ for a point $x \in X$, where $U \subset X$ is a neighbourhood of x and f is a continuous function on U. Let $V \subset X$ be a small (simply connected) open neighbourhood of $f(x)$, such that there exists a continuous square root function on V. Then $(f^{-1}(V), \sqrt{f})$ is an element of $\mathscr{F}_x$ and $\varphi_x([f^{-1}(V), \sqrt{f}]) = [f^{-1}(V), f] = [U, f]$.