D-MATH | Algebraic Geometry | FS 2019
Prof. Emmanuel Kowalski

Solutions Sheet 4

Sheaves & Schemes

1. Let X be a topological space and $\mathcal{F}$ a presheaf on X. We denote by $\mathcal{F}^s$ the sheaf associated to $\mathcal{F}$. Show that the canonical map

$$\theta : \mathcal{F} \to \mathcal{F}^s,$$

induces an isomorphism

$$\theta : \mathcal{F}_x \to \mathcal{F}^s_x,$$

for all $x \in X$.

Solution: We recall that for any $U \subset X$ open, one defines $\mathcal{F}^s(U)$ as the set of function

$$s : U \to \bigcup_{x \in U} \mathcal{F}_x,$$

such that for each $x \in U$ there is a neighborhood $V \subset U$ of x and an element $t \in \mathcal{F}(V)$ such that for all $y \in V$, the germ t_y of t at y is equal to $s(y)$. Moreover, one has the canonical map θ defined as follows

$$\theta : \begin{array}{ccc} \mathcal{F}(U) & \to & \mathcal{F}^s(U) \\ w & \mapsto & \theta(w) : y \mapsto w_y \end{array}$$

which induces the map

$$\theta_x : \begin{array}{ccc} \mathcal{F}_x & \to & \mathcal{F}^s_x(U) \\ (w, U) & \mapsto & (\theta(w) : U) \end{array}$$

for any $x \in X$. This map is clearly injective. Let us check surjectivity. Let $(s, U) \in \mathcal{F}^s_x$, then by definition there is a neighborhood $V \subset U$ of x and an element $t \in \mathcal{F}(V)$ such that for all $y \in V$, the germ t_y of t at y is equal to $s(y)$. Thus $\theta_x(t, V) = (\theta(t), V) = (s_{|V}, V) = (s, U)$ as we wanted.

2. Let X be a topological space and $\mathcal{F}$, $\mathcal{G}$ sheaves of abelian groups on X. In the following $U \subset X$ is always an open subset of X.

   (a) If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (of abelian groups), then $\mathrm{Ker}(f) : U \to \mathrm{Ker}(f_U)$,is a sheaf (with the obvious restriction).

   (b) If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves (of abelian groups), then $\mathcal{H} : U \to \mathrm{Im}(f_U)$, is not a sheaf in general. Let us denote $\mathrm{Im}(f) := \mathcal{H}^s$. Show that there is an injective morphism

$$g : \mathrm{Im}(f) \hookrightarrow \mathcal{G}.$$

Solution:

(a) It is clear that $\mathrm{Ker}(f)$ is a presheaf. Let $U \subset X$ be an open subset of and $\{U_i\}_{i \in I}$ be a covering of U. We first check the locality condition: assume that $s \in \mathrm{Ker}(f_U)$ such that $s_{|U_i} = 0$ for a any $i \in I$. Then it follows that $s = 0$ since $\mathrm{Ker}(f_U) \subset \mathcal{F}(U)$ and $\mathcal{F}$ is a sheaf. Now let us check the gluing condition: let $\{s_i \in \mathrm{Ker}(f_{U_i})\}_{i \in I}$ such that for any $i, j \in I$, $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$. Since $\mathcal{F}$ is a sheaf, there exists $s \in \mathcal{F}(U)$ such that $s_{|U_i} = s_i$ for any $i \in I$. On the other hand, $(f_U(s))_{|U_i} = f_{U_i}(s_i) = 0$. Thus $f_U(s) = 0$ since $\mathcal{F}$ is a sheaf. Hence, $s \in \mathrm{Ker}(f_U)$.

(b) Let us consider $X := \mathbb{C} \setminus \{0\}$ with standard topology, $\mathscr{F} = \mathscr{G}$ the sheaf of nowhere-zero continuous complex-valued functions and let $f : \mathscr{F} \to \mathscr{G}$ be the morphism that sends a function φ to φ^2. We have seen in the previous exercise sheet that for every point $x \in X$ the induced morphism on the stalks f_x is surjective. By contradiction, let us assume that $\mathcal{H} : U \to \mathrm{Im}(f_U)$ is a sheaf. This would imply that there exists a continuous root function map $z \mapsto \sqrt{z}$ on $\mathbb{C} \setminus \{0\}$ but this is not possible. For the second part of the exercise, we know that $i : \mathcal{H} \hookrightarrow \mathcal{G}$ is a morphism of presheaf by definition. Then using the universal property of the associated sheaf $(\mathcal{H}^s = \mathrm{Im}(f), \theta)$, we get the commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \xhookrightarrow{\ i\ } & \mathcal{G} \\ \downarrow \theta & \nearrow g & \\ \mathrm{Im}(f) & & \end{array} .$$

This leads to the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}_x & \xhookrightarrow{\ i_x\ } & \mathcal{G}_x \\ \downarrow \theta_x & \nearrow g_x & \\ \mathrm{Im}(f) & & \end{array} ,$$

for any $x \in X$. On the other hand, θ_x is an isomorphism for any $x \in X$ thanks to the previous exercise. Then g_x is injective for any $x \in X$ and this implies that g is injective as we wanted.

3. *Inverse Image Sheaf.* Let $f : X \to Y$ be a continuous map of topological spaces. For a sheaf $\mathscr{G}$ of abelian groups on Y we define the *inverse image* sheaf $f^{-1}\mathscr{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \supset f(U)} \mathscr{G}(V)$, where U is any open set of X and the direct limit (see exercise A) is taken over all open subsets V of Y containing $f(U)$. Prove that for every sheaf $\mathscr{F}$ on X there is a natural map

$f^{-1}f_*\mathscr{F} \to \mathscr{F}$ and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Prove that this induces a natural bijection of sets

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y. One says that f^{-1} and f_* are adjoint functors.

Solution: Let $\mathscr{F}$ be a sheaf on X. Denote $\mathscr{F}'$ for the presheaf $U \mapsto \lim_{V \supset f(U)}(f_*\mathscr{F})(V)$. For every $V \supset f(U)$ we have that $f^{-1}(V) \supset f^{-1}f(U) \supset U$ and since by definition $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ we conclude that there is a restriction map $(f_*\mathscr{F})(V) \to \mathscr{F}(U)$. Hence we get a morphism $\mathscr{F}' \to \mathscr{F}$ of presheaves. Since $f^{-1}f_*\mathscr{F}$ is the sheaf associated to $\mathscr{F}'$ and by the universal property of the associated sheaf we get a morphism of sheaves $f^{-1}f_*\mathscr{F} \to \mathscr{F}$.

Now let $\mathscr{G}$ be a sheaf on Y. Denote by $\mathscr{G}'$ the presheaf $U' \mapsto \lim_{V \supset f(U')}\mathscr{G}(V)$ on X. We have $U \supset ff^{-1}(U)$ and thus we get a map $\mathscr{G}(U) \to \mathscr{G}'(f^{-1}(U))$. Composing this with the sheafification map $\mathscr{G}'(f^{-1}(U) \to (f^{-1}\mathscr{G})(f^{-1}(U))$ we get a morphism of sheaves $\mathscr{G} \to f_*f^{-1}\mathscr{G}$.

Let $\varphi : f^{-1}\mathscr{G} \to \mathscr{F}$ be a morphism of sheaves on X. Using the previous construction we get a morphism $\mathscr{G} \to f_*f^{-1}\mathscr{G} \to f_*\mathscr{F}$. Conversely, every morphism $\psi : \mathscr{G} \to f_*\mathscr{F}$ of sheaves on X induces a morphism $f^{-1}\mathscr{G} \to f^{-1}f_*\mathscr{F} \to \mathscr{F}$. To see that these two constructions are inverse to each other, either trace through the individual maps and note that they are only restriction maps and inclusions to the direct limit, or look at the map induced on the stalks.

4. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. Prove that if f is a homeomorphism and $f^\sharp$ is an isomorphism, then $(f, f^\sharp)$ is an isomorphism.

   Solution: We need to construct an inverse morphism $(g, g^\sharp) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally ringed spaces. Since f is a homeomorphism, we can define $g := f^{-1}$. On sheaves however, the definition is not so clear, since $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$. For every open subset $U \subset X$ we define the homomorphism

$$g^\sharp(U) : \mathcal{O}_X(U) \to g_*\mathcal{O}_Y(U) = \mathcal{O}_Y(g^{-1}(U)) = \mathcal{O}_Y(f(U))$$

   as

$$f^{\sharp-1}(f(U)) : \mathcal{O}_X(f^{-1}f(U)) = \mathcal{O}_X(U) \to \mathcal{O}_Y(f(U)).$$

   This is well-defined and one sees that it provides an inverse of $(f, f^\sharp)$ as morphism of locally ringed spaces.

5. Let A be a ring and set $X := \mathrm{Spec}(A)$. Let $f \in A$ and let $U_f \subset X$ be the open complement of $V((f))$.

   (a) Show that the locally ringed space $(U_f, \mathcal{O}|_{U_f})$ is isomorphic to $\mathrm{Spec}(A_f)$.

(b) For another element $g \in A$ describe the restriction map $\mathcal{O}(U_f) \to \mathcal{O}(U_{fg})$ in terms of a ring homomorphism $A_f \to A_{fg}$.

Solution:

(a) Denote $Y := \operatorname{Spec}(A_f)$. The prime ideals of A which do not contain (f) are in one-to-one correspondence with prime ideals in A_f. This correspondence is given as the pullback of the localisation map $\varphi : A \to A_f$. This pullback preserves inclusions and thus we conclude that Y and U_f are homeomorphic as topological spaces via φ^*. Furthermore, for every point $\mathfrak{p} \in U_f \cong Y$ we have $(\mathcal{O}|_{U_f})_{\mathfrak{p}} = A_{\mathfrak{p}} \cong (A_f)_{\varphi(\mathfrak{p})} = \mathcal{O}_{Y,\mathfrak{p}}$ and the isomorphism is given by $\varphi_{\mathfrak{p}}$. Since a morphism of sheaves is an isomorphism if and only if it is an isomorphism on the stalks, we conclude that $\mathcal{O}|_{U_f} \cong \mathcal{O}_Y$ as sheaves. We conclude that $\operatorname{Spec}(A_f) \cong (U_f, \mathcal{O}|_{U_f})$ as locally ringed spaces via the isomorphism φ^*.

(b) The restriction is just the localisation map $A_f \to (A_f)_g \cong A_{fg}$. To see this, recall the proof of the isomorphism $A_f \cong \mathcal{O}(U_f)$. There we have proved that the map sending an element $a/f^n \in A_f$ to the section taking a point $\mathfrak{p} \in \mathcal{O}(U_f)$ to $a/f^n \in A_{\mathfrak{p}}$ is an isomorphism. We see that we have a commutative diagram:

$$\begin{array}{ccc} A_f & \xrightarrow{\cong} & \mathcal{O}(U_f) \\ \downarrow & & \downarrow \\ A_{fg} & \xrightarrow{\cong} & \mathcal{O}(U_{fg}) \end{array}$$

where the left vertical map is the localisation map and the right vertical map is the restriction.

6. Consider $S_1 := \operatorname{Spec}(\mathbb{Q}[X,Y]/(XY))$ and $S_2 := \operatorname{Spec}(\mathbb{Q}[X,Y]/(X^2+Y^2))$.

(a) Compute $\operatorname{Hom}_{\mathrm{Sch}}(\operatorname{Spec}(\mathbb{Q}), S_i)$ for $i = 1, 2$.

(b) Deduce that S_1 and S_2 are not isomorphic schemes.

(c) Let $S_1' := \operatorname{Spec}(\mathbb{Q}(i)[X,Y]/(XY))$ and $S_2' := \operatorname{Spec}(\mathbb{Q}(i)[X,Y]/(X^2+Y^2))$. Prove that $S_1' \cong S_2'$ as schemes.

Solution:

(a) We have

$$\begin{aligned} \operatorname{Hom}_{\mathrm{Sch}}(\operatorname{Spec}(\mathbb{Q}), S_1) &\cong \operatorname{Hom}_{\mathrm{Ring}}(\mathbb{Q}[X,Y]/(XY), \mathbb{Q}) \\ &= \{(a,b) \in \mathbb{Q} \mid ab = 0\} = (\mathbb{Q} \times \{0\}) \cup (\{0\} \times \mathbb{Q}) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Hom}_{\mathrm{Sch}}(\operatorname{Spec}(\mathbb{Q}), S_2) &\cong \operatorname{Hom}_{\mathrm{Ring}}(\mathbb{Q}[X,Y]/(X^2+Y^2), \mathbb{Q}) \\ &= \left\{(a,b) \in \mathbb{Q} \;\middle|\; a^2 + b^2 = 0\right\} = \{(0,0)\} \end{aligned}$$

(b) Since these two Hom-sets are not isomorphic (different cardinality), the schemes S_1 and S_2 can not be isomorphic.

(c) To give an isomorphism of schemes $S_2' \to S_1'$ it is enough to give an isomorphism of rings $\mathbb{Q}(i)[X,Y]/(XY) \to \mathbb{Q}(i)[X,Y]/(X^2+Y^2)$. We define this ring homomorphism by $X \mapsto X+iY$ and $Y \mapsto X-iY$. Indeed $XY \mapsto X^2+Y^2$ and so the homomorphism is well-defined. Furthermore the homomorphism defined by $\frac{1}{2}(X+Y) \leftarrow\!\shortmid X$ and $\frac{i}{2}(Y-X) \leftarrow\!\shortmid Y$ provides a both-sided inverse, hence it is an isomorphism.

7. Let X be a scheme. For any point $x \in X$ we define the Zariski tangent space T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k and let $k[\varepsilon]/(\varepsilon^2)$ be the *ring of dual numbers* over k. Show that to give a morphism of schemes over k of $\operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))$ to X is equivalent to giving a point $x \in X$ such that $k(x) = k$ and an element of T_x.

Solution: Let $\varphi : \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)) \to X$ be a k-morphism. The ring $A := k[\varepsilon]/(\varepsilon^2)$ has only one prime ideal (ε), hence $\operatorname{Spec}(A)$ is only one point, say $\mathfrak{o}$. Thus we get a point $x := \varphi(\mathfrak{o}) \in X$. The morphism φ induces a ring homomorphism $\mathcal{O}_{X,x} \to A$ with $\varphi(\mathfrak{m}_x) \subset (\varepsilon)$. We can thus define a k-linear map $\mathfrak{m}_x \to A \to k$ given by $a \mapsto \frac{a}{\varepsilon} \mapsto \frac{a}{\varepsilon} \mod (\varepsilon)$, whose kernel is $\mathfrak{m}_x^2$. Thus we have a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$, i.e. a tangent vector.

Conversely, suppose that we are given a point $x \in X$ and a tangent vector $f \in T_x$. Every element in the stalk $\mathcal{O}_{X,x}$ is uniquely given by a sum $a+b$ with $a \in k$ and $b \in \mathfrak{m}_x$. We define $\varphi(a+b) := a + f(b)\varepsilon \in A$. This defines a ring homomorphism $\mathcal{O}_{X,x} \to A$. This gives us a scheme homomorphism $\operatorname{Spec}(A) \to X$ by sending the point to x and on the sheaves we have $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to A$ for $x \in U$ and $\mathcal{O}_X(U) \to 0$ if $x \notin U$.