# Solutions Sheet 5

## Schemes

1. Let $X$ be a scheme and $f \in \mathcal{O}_X(X)$ a global section. Define $X_f$ to be the subset of points $x \in X$ such that the germ $f_x$ of $f$ at $x$ is not contained in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

   (a) If $U = \operatorname{Spec}(B)$ is an open affine subscheme of $X$ and if $\bar{f} \in B = \mathcal{O}_U(U)$ is the restriction of $f$, show that $U \cap X_f = D(\bar{f})$. Conclude that $X_f$ is an open subset of $X$.

   (b) Assume that $X$ is quasi-compact. Let $A := \mathcal{O}_X(X)$ and let $a \in A$ be an element whose restriction to $X_f$ is 0. Show that there exists an integer $n > 0$ such that $f^n a = 0$.

   (c) Now assume that $X$ has a finite cover by open affines $U_i$ such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \mathcal{O}_{X_f}(X_f)$. Show that there exists an integer $n > 0$ such that $f^n b$ is the restriction of an element of $A$.

   (d) With the hypothesis of (c) conclude that $\mathcal{O}_{X_f}(X_f) \cong A_f$.

   *Solution*:

   (a) We have that

   $$\begin{aligned} X_f \cap U :&= \{\mathfrak{p} \in \operatorname{Spec}(B) : f_{\mathfrak{p}} \notin \mathfrak{p}B_{\mathfrak{p}}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec}(B) : \overline{f} \notin \mathfrak{p}B_{\mathfrak{p}}\} \\ &= \operatorname{Spec}(B_{\overline{f}}), \end{aligned}$$

   thus $X_f \cap U = D(\overline{f})$ has we wanted.

   (b) Since $X$ is quasi-compact by hypothesis, we can find a finite affine cover, $\{U_i\}_{i=1}^n$, of $X$. For any $i = 1, ..., n$, let $B_i$ denote the ring such that $U_i = \operatorname{Spec}(B_i)$. Let $a \in \mathcal{O}_X(X)$ such that $a_{|X_f} = 0$. For any $i = 1, ..., n$ we have that $a_{|X_f \cap U_i} = 0$. On the other hand by part $(a)$ we know that $X_f \cap U_i = (U_i)_{f_i} = \operatorname{Spec}(B^i_{f_i})$, where $f_i$ denote the restriction of $f$ to $U_i$. Thus for $i = 1, ..., n$, $a_{|(U_i)_{f_i}} = 0$ and this implies that there exist $n_i \geqslant 0$ such that

   $$f_i^{n_i} \cdot a_{|U_i} = 0 \text{ in } B_i.$$

   Let us take $N := \max_{1 \leqslant i \leqslant n} n_i$. For any $i = 1, ..., n$ one has that $f^N a_{|U_i} = f_i^N \cdot a_{|U_i} = 0$. Since $\{U_i\}_{i=1}^n$ is a cover of $X$ and $\mathcal{O}_X$ is a sheaf we conclude that $f^N a = 0$.

(c) Let $b \in \mathcal{O}_{X_f}(X_f)$, $\{U_i\}_{i=1}^n$ be a finite affine cover such that for any $i, j$, $U_i \cap U_j$ is quasi-compact and for any $i = 1, ..., n$, let $b_i$ denotes the restriction of $b$ to $X_f \cap U_i = (U_i)_{f_i}$. Since $(U_i)_{f_i} = \mathrm{Spec}((B_i)_{f_i})$, there exists $n_i \geqslant 0$ such that $f_i^{n_i} b_i \in B_i$. Let us take $N := \max_{1 \leqslant i \leqslant n} n_i$. Fix $i, j$ and consider

$$b_{i,j} := (f_i^N b_i)_{|U_i \cap U_j} - (f_j^N b_j)_{|U_i \cap U_j} \in \mathcal{O}_X(U_i \cap U_j).$$

It is easy to see that $b_{ij||U_i \cap U_j \cap X_f} = 0$, then we can apply part $(b)$ since $U_i \cap U_j$ is quasi compact. Thus there exists $n_{i,j} \geqslant 0$ such that

$$f_{|U_i \cap U_j}^{n_{i,j}} c_{i,j} = 0.$$

Let us take $M := \max_{i,j} n_{i,j}$, We claim that $f^{N+M} b \in \mathcal{O}_X(X)$. For any $i$, $f_i^{N+M} b_i \in \mathcal{O}_X(U_i)$, such that $(f_i^{N+M} b_i)_{|U_i \cap U_j} = (f_j^{N+M} b_j)_{|U_i \cap U_j}$ for any $i, j$. Thus $f^{N+M} b \in \mathcal{O}_X(X)$ since $\mathcal{O}_X$ is a sheaf.

(d) The inclusion $A_f \subset \mathcal{O}_{X_f}(X_f)$ follows because for any $i = 1, ..., n$, $\{X_f \cap U_i\}_{i=1}^n$ is a covering of open affine subset of $X_f$ and $\mathcal{O}_{X_f}(X_f \cap U_i) = B_f^i$. For the other inclusion, let $s \in \mathcal{O}_{X_f}(X_f)$ then by part $(c)$ there exists $n \geqslant 0$ such that $a := f^n s \in \mathcal{O}_X(X)$. On the other hand $f$ is invertible in $\mathcal{O}_{X_f}(X_f)$, thus $s = \frac{a}{f^n}$ as we wanted.

2. *A Criterion for Affineness.*

(a) Let $f : X \to Y$ be a morphism of schemes and suppose that $Y$ can be covered by open subsets $U_i$ such that for each $i$, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then $f$ is an isomorphism.

(b) A scheme $X$ is affine if and only if there is a finite set of elements $f_1, \ldots, f_r \in A := \mathcal{O}_X(X)$ such that the open subsets $X_{f_i}$ defined in exercise 1 are affine and $f_1, \ldots, f_r$ generate the unit ideal.

*Solution*:

(a) Let $f : X \to Y$ a map as in the statement of the exercise. To check that $f$ is an isomorphis it is enough to check that for any $x \in X$ the map

$$f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is an isomorphism. Let $U_i$ be an open set in the covering such that $x \in U_i$, then we have that

$$(f_{|U_i})_x : (\mathcal{O}_{Y|_{f(U_i)}})_{f(x)} \to (\mathcal{O}_{X|_{U_i}})_x$$

is an isomorphism by hypothesis. On the other hand we have $(\mathcal{O}_{Y|_{f(U_i)}})_{f(x)} \cong \mathcal{O}_{Y,f(x)}$ and $(\mathcal{O}_{X|_{U_i}})_x \cong \mathcal{O}_{X,x}$ since $U_i$ and $f(U_i)$ are open subset respectively of $X$ and $Y$.

(b) First We prove that $\cup_{i=1}^{n} X_{f_i}$. By contradiction, assume that there exists $x \in X \setminus \cup_{i=1}^{n} X_{f_i}$, then $f_i \in \mathfrak{m}_x$ for any $i = 1, ..., n$. This would implies that $1 \in \mathfrak{m}_x$ and this is not possible since $\mathcal{O}_{X,x}$ has to be a local ring. Because we are assuming $X_{f_i}$ to be affine we can apply part $(d)$ of the previous exercise getting $X_{f_i} = \mathrm{Spec}(A_{f_i})$ for any $i = 1, ..., n$. From the ring homomorphisms

$$\begin{array}{rcl} g_i^* : A & \to & A_{f_i} \\ a & \mapsto & \frac{a}{1}, \end{array}$$

we get the scheme morphisms

$$g_i : X_{f_i} \to \mathrm{Spec}(A)$$

which are isomorphisms between $X_f$ and $g_i(X_{f_i}) = \mathrm{Spec}(A_{f_i})$. Gluing together these maps we obtain a morphism

$$g : X \to \mathrm{Spec}(A).$$

Then we conclude applying part $(a)$.

3. Let us recall that a *graded ring* is a ring $S$ together with a decomposition $S = \bigoplus_{d \geqslant 0} S_d$ of $S$ into a direct sum of abelian groups $S_d$, such that for any $d, e \geqslant 0$, $S_d \dot{S}_e \subset S_{de}$. An element of $S_d$ is called a *homogeneous element of degree d*. An ideal $I \subset S$ is a *homogeneous ideal* if $I = \bigoplus_{d \geqslant 0} (I \cap S_d)$.

(a) Let $K$ be an algebraically closed field, and let $f_1, ..., f_m \in K[X_0, ..., X_n] = S$ be homogeneous polynomials. Re Let $I = (f_1, ... f_m) \subset S$ prove that

*i)* The ideal $I$ is an homogeneous ideal.

*ii)* Show that $B := S/I$ is a graded ring.

(b) Let $B_+ = I = \bigoplus_{d \geqslant 1} B_d$. One considers

$$\mathrm{Proj}(B) := \{\mathfrak{p} \subset B \text{ homogeneous prime ideals} : B_+ \not\subseteq \mathfrak{p}\}.$$

For any homogeneous ideal, $I$, one defines

$$V(I) := \{\mathfrak{q} \in \mathrm{Proj}(B) : I \subset \mathfrak{q}\}.$$

Show that these sets form the closed sets of a topology on $\mathrm{Proj}(B)$.

(c) We define a sheaf of rings $\mathcal{O}$ on $\mathrm{Proj}(S)$ as follows: For each $\mathfrak{p} \in \mathrm{Proj}(S)$ we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where $T$ is the multiplicative system of all homogeneous elments of $S$ which are not in $\mathfrak{p}$. For any open subset $U \subset \mathrm{Proj}(S)$ we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$ we have $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and there is an open neighbourhood $V$ of $\mathfrak{p}$ in $U$ and homogeneous elements $a, f$ in $S$ of the same degree such that for all $\mathfrak{q} \in V$ we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. Prove the following:

*i)* For any $\mathfrak{p} \in \text{Proj}(S)$, the stalk $\mathcal{O}_\mathfrak{p}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.

*ii)* For any homogeneous $f \in S_+$ let $U_f^+$ be the complement of $V((f))$. These open sets cover $\text{Proj}(S)$ and there is an isomorphism of locally ringed spaces

$$(U_f^+, \mathcal{O}|_{U_f^+}) \cong \text{Spec}(S_{(f)})$$

where $S_{(f)}$ is the subring of elements of degree zero in the localized ring $S_f$.

(d) Conclude that $(\text{Proj}(B), \mathcal{O})$ is a scheme.

*Solution*:

(a) Let $f_1, ..., f_m \in K[X_0, ..., X_n] = S$ be homogeneous polynomials and denote $I = (f_1, ..., f_m)$. Then

*i)* It is clear that $I \supset \bigoplus_{d \geqslant 0}(I \cap S_d)$. Let us prove the other inclusion. Let $f \in I$, then there exist $a_1, ..., a_m \in K[X_0, ..., X_n]$ such that

$$f = \sum_{i=1}^{m} a_i f_i.$$

For any $i$ we can decompose $a_i$ as sums of monomial

$$a_i = \sum_{d=1}^{\deg a_i} \sum_{\substack{d_0, \dots, d_n \\ d_0 + \dots + d_n = d}} a_{d_0, \dots, d_n} X_0^{d_0} \cdots X_n^{d_n},$$

where $a_{d_0,...,d_n} \in K$ for any $d_0, ..., d_n$. On the other hand, for any $d_0, ..., d_n$ such that $d_0 + \cdots + d_n = d$, $f_i \cdot (X_0^{d_0} \cdots X_n^{d_n}) \in I \cap S_{d + \deg f_i}$. Thus, $a_i f_i \in \bigoplus_{d \geqslant 0}(I \cap S_d)$ and this implies that $f \in \bigoplus_{d \geqslant 0}(I \cap S_d)$ as we wanted.

*ii)* Since $S = \bigoplus_{d \geqslant 0} S_d$ and $I = \bigoplus_{d \geqslant 0}(I \cap S_d)$ it follows that $B = \bigoplus_{d \geqslant 0} B_d$ where $B_d := S_d/(IcapS_d))$. Moreover, $B_d B_e \subset B_{de}$ since $S_d S_e \subset S_{de}$. Thus $B$ is a graded ring.

(b) This is analogous to the proof for the case of affine schemes.

(c) For each $\mathfrak{p} \in \text{Proj}(S)$ we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where $T$ is the multiplicative system of all homogeneous elments of $S$ which are not in $\mathfrak{p}$. For any open subset $U \subset \text{Proj}(S)$ we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$ we have $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and there is an open neighbourhood $V$ of $\mathfrak{p}$ in $U$ and homogeneous elements $a, f$ in $S$ of the same degree such that for all $\mathfrak{q} \in V$ we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. Then *i)* As in the affine case, we define a homomorphism $\varphi : \mathcal{O}_\mathfrak{p} \to S_{(\mathfrak{p})}$ by sending any local section $s$ in a neighbourhood of $\mathfrak{p}$ to its value $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$. This is surjective, since every element $a/f \in S_{(\mathfrak{p})}$ for two homogeneous elements $a, f \in S$ of the same degree such that $f \notin \mathfrak{p}$, define a well-defined section $U_f \to \coprod S_{(\mathfrak{p})}$, whose image at $\mathfrak{p}$ is $a/f$. The map $\varphi$ is also injective, because for any element $s_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}$ such that $\varphi(s_\mathfrak{p}) = 0$ there is an open neighbourhood $U$ of $\mathfrak{p}$ and a section $s$ over $U$ such that $s(\mathfrak{p}) = s_\mathfrak{p}$ and $s$ is given by $\mathfrak{q} \mapsto a/f$ for some homogeneous elements $a, f \in S$ of the same degree and $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in U$. Since $\varphi(s_\mathfrak{p}) = 0$ we conclude that there is a homogeneous element $u \notin \mathfrak{p}$ such that $ua = 0$. Thus $s|_{U_u \cap U} = 0$, which implies that $s_\mathfrak{p} = 0$. Hence $\varphi$ is an isomorphism.

*ii)* Since $\mathrm{Proj}(S)$ is the set of all homogeneous prime ideals which do not contain $S_+$, the sets $U_f^+$ for homogeneous $f \in S_+$ cover $\mathrm{Proj}(S)$. Consider the localisation map $S \to S_f$. We know that $S_{(f)}$ is a subring of $S_f$. For a homogeneous ideal $\mathfrak{a} \subset S$ define $\varphi(\mathfrak{a}) := (\mathfrak{a}S_f) \cap S_{(f)}$. In particular for $\mathfrak{p} \in U_f^+$ we have $\varphi(\mathfrak{p}) \in \mathrm{Spec}(S_{(f)})$. This respects inclusions and is bijective, and so $\varphi$ defines a homeomorphism $\varphi : U_f^+ \to \mathrm{Spec}(S_{(f)})$. Note that for $\mathfrak{p} \in U_f^+$ we have $S_{(\mathfrak{p})} \cong (S_{(f)})_{(\varphi(\mathfrak{p}))}$ and so the sheaf homomorphism is an isomorphism. This proves that $(U_f^+, \mathcal{O}|_{U_f^+}) \cong \mathrm{Spec}(S_{(f)})$.