Solutions Sheet 4

Schemes

1. *Double Points.* Let k be a field and $Y \hookrightarrow \mathbb{A}^2_k$ be a closed subscheme with image containing the origin $(0,0)$ in $\mathbb{A}^2_k$ and such that $\mathcal{O}_Y(Y) \cong k[\varepsilon]/(\varepsilon^2)$. Denote by $\varphi : k[x,y] \to \mathcal{O}_Y(Y)$ the surjection defining the inclusion $Y \hookrightarrow \mathbb{A}^2$. Prove that the kernel of φ contains a non-zero element $\alpha x + \beta y$ for some $\alpha, \beta \in k$. Write $X_{\alpha,\beta} := \mathrm{Spec}(k[x,y]/\ker(\varphi))$ and show that $X_{\alpha,\beta}$ can also be characterized as the composition of the natural morphism $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)) \to \mathrm{Spec}(k[\varepsilon]) \cong \mathbb{A}^1_k$ with the inclusion of the line $\mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k$ given by $x \mapsto (\beta x, -\alpha x)$.

   Solution: You find this solution in the Book of Eisenbud and Harris, Examples $II.9$, $II.10$.

2. Let k be an algebraically closed field and let $Z := \mathrm{Spec}(k[X_1, \dots, X_n]/I) \subset \mathbb{A}^n_k$ be a closed subscheme of dimension 0 supported at the origin (i.e. $\sqrt{I} = (X_1, ..., X_n)$). Furthermore, suppose that $k[X_1, \dots, X_n]/I$ is a 3-dimensional k-vector space. Prove that Z is isomorphic to either $A := \mathrm{Spec}(k[X]/(X^3))$ or to $B := \mathrm{Spec}(k[X,Y]/(X^2, XY, Y^2))$ and that A and B are not isomorphic to each other.

   Solution: Let us start proving that A and B are not isomorphic. To show this it is enough to prove that $k[X]/(X^3)$ and $k[X,Y]/(X^2,XY,Y^2)$ are not isomorphic as rings. Let $\varphi : k[X]/(X^3) \to k[X,Y]/(X^2,XY,Y^2)$ be any ring homomorphism from $k[X]/(X^3)$ to $k[X,Y]/(X^2,XY,Y^2)$. Then $\varphi(X^2) = \varphi(X)^2 = 0$. On the other hand $X^2 \neq 0$ in A, thus $\mathrm{Ker}(\varphi) \neq 0$. It follows that $k[X]/(X^3)$ and $k[X,Y]/(X^2,XY,Y^2)$ can not be isomorphic since any homorphism from from $k[X]/(X^3)$ to $k[X,Y]/(X^2,XY,Y^2)$ is not injective.

   Let us prove the other part of the Exercise. in the following, x_i denotes the imagine of X_i in $k[X_1, \dots, X_n]/I$. Since $\sqrt{I} = (X_1, ..., X_n)$, then $\mathrm{Spec}(k[X_1, \dots, X_n]/I) = \{\mathfrak{m}\}$ where $\mathfrak{m}$ is the reduction of $(X_1, ..., X_n)$ modulo I. Moreover we have that

   $$k[X_1, \dots, X_n]/I = k \oplus \mathfrak{m}$$

   thus $\dim_k(\mathfrak{m}) = 2$. We claim that $\mathfrak{m}^3 = 0$. Let us consider the chain

   $$\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3.$$

   Then $\mathfrak{m} \supsetneq \mathfrak{m}^2$, otherwise by Nakayama's Lemma $\mathfrak{m} = 0$. Thus $\dim_k(\mathfrak{m}^2) \leqslant 1$ If $\dim_k(\mathfrak{m}^2) = 0$ we have done, otherwise $\mathfrak{m}^2 \supsetneq \mathfrak{m}^3$ (again thanks to Nakayama'sd Lemma) and then $\mathfrak{m}^3 = 0$. Now we have to distinguish to situation *i)* $\mathfrak{m}^2 = 0$. In this case we have that $\mathfrak{m} = \text{span}_k\{x_1, ..., x_n\}$. Because $\dim_k(\mathfrak{m}) = 2$, without loss generalities we may assume that $\text{span}\{x_1, x_2\}$. On the other hand $\mathfrak{m}^2 = 0$ implies that $x_1^2 = x_1x_2 = x_2^2 = 0$. Hence the map

$$\begin{array}{ccc} k[X,Y]/(X^2, Y^2, XY) & \to & k[X_1, ..., X_n]/I \\ X & \mapsto & x_1 \\ Y & \mapsto & x_2, \end{array}$$

is a ring isomorphism and so $Z \cong B$.

ii) $\mathfrak{m}^2 \neq 0$. We claim that there exists $i \in \{1, ..., n\}$ such that $x_i^2 \neq 0$. By contradiction assume this is not the case. Let x_i, x_j with $i \neq j$ such that $x_ix_j \neq 0$ (we know that there exists at least one of these pairs because $\mathfrak{m}^2 \neq 0$ and we are assuming $x_i^2 = 0$ for any $i = 1, ..., n$). Then $\text{span}_k\{x_i, x_ix_j\} = \mathfrak{m}$: let $\alpha, \beta \in k$ such that

$$\alpha x_i + \beta x_i x_j = 0,$$

then

$$x_i(\alpha + \beta x_i x_j) = 0.$$

This implies $\alpha = 0$, otherwise $\alpha + \beta x_i x_j$ is invertible in $k[X_1, \ldots, X_n]/I$ and so $x_i = 0$ and this is not possible since we are assuming $x_ix_j = 0$. On the other hand $\alpha = 0$ implies $\beta = 0$ again because $x_ix_j \neq 0$. Then $\dim(\text{span}_k\{x_i, x_ix_j\}) = 2$ and so $\text{span}_k\{x_i, x_ix_j\} = \mathfrak{m}$. In particular there exist $\alpha, \beta \in k$ such that

$$x_j = \alpha x_i + \beta x_i x_j.$$

Multiplying both sides by x_j we get

$$x_j^2 = \alpha x_i x_j + \beta x_i x_j^2,$$

thus $\alpha x_i x_j = 0$ which implies $\alpha = 0$. So we have

$$x_j = \beta x_i x_j.$$

implies $x_j(1 - \beta x_i) = 0$. On the other hand $1 - \beta x_i$ is invertible in $k[X_1, \ldots, X_n]/I$, thus $x_j = 0$ and then $x_ix_j = 0$ and this is absurd since we are assuming $x_ix_j \neq 0$. Hence, without loss of generality we may assume that x_1 is such that $x_1^2 \neq 0$. Then $\text{span}_k\{x_1, x_1^2\} = \mathfrak{m}$: indeed if $x_1^2 = \lambda x_1$ for some $\lambda \in k^\times$ then $x_1^3 = \lambda^2 x_1 \neq 0$ since $x_1 \neq 0$. On the other hand $x_1^3 \in \mathfrak{m}^3 = 0$ and this leads to a contradiction. Thus $x_1^2 \neq 0$ and $x_1^2 \notin \text{span}_k\{x_1\}$, i.e. $\text{span}_k\{x_1, x_1^2\} = \mathfrak{m}$. Hence the map

$$\begin{array}{ccc} k[X]/(X^3) & \to & k[X_1, ..., X_n]/I \\ X & \mapsto & x_1 \end{array}$$

is a ring isomorphism and so $Z \cong A$.

Let us consider $V = \text{span}\{x_1, ..., x_n\}$, we have two possibilities:

i) $\dim_k(V) = 2$. With out loss of generalities we may assume $V = \text{span}\{x_1, x_2\}$. Then $k[X_1, \ldots, X_n]/I = \text{span}\{1, x_1, x_2\}$

ii) $\dim_k(V) = 1$

3. Let $X := \mathbb{A}^2_{\mathbb{C}} \setminus \{0\} \subset \mathbb{A}^2_{\mathbb{C}}$. Prove:

   (a) The restriction map $\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}(\mathbb{A}^2_{\mathbb{C}}) \to \mathcal{O}_X(X)$ is an isomorphism.

   (b) The scheme X is not an affine scheme.

   Solution:

   (a) We compute $\Gamma(\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}})$. Since

$$\mathbb{A}^2_{\mathbb{C}} \setminus \{0\} = \mathbb{A}^2_{\mathbb{C},X} \cup \mathbb{A}^2_{\mathbb{C},Y}$$

   where

$$\mathbb{A}^2_{\mathbb{C},X} := \mathbb{A}^2_{\mathbb{C}} \setminus \{\mathfrak{p} : (X) \subset \mathfrak{p}\}, \quad \mathbb{A}^2_{\mathbb{C},Y} := \mathbb{A}^2_{\mathbb{C}} \setminus \{\mathfrak{p} : (Y) \subset \mathfrak{p}\},$$

   we have that $\Gamma(\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}})$ is the kernel of the map

$$\begin{array}{ccc} \Gamma(\mathbb{A}^2_{\mathbb{C},X}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) \oplus \Gamma(\mathbb{A}^2_{\mathbb{C},Y}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) & \to & \Gamma(\mathbb{A}^2_{\mathbb{C},XY}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) \\ (s,t) & \mapsto & s_{|_{\mathbb{A}^2_{\mathbb{C},XY}}} - t_{|_{\mathbb{A}^2_{\mathbb{C},XY}}}, \end{array}$$

   where $\mathbb{A}^2_{\mathbb{C},XY} = \mathbb{A}^2_{\mathbb{C},X} \cap \mathbb{A}^2_{\mathbb{C},Y}$. Let $s \in \Gamma(\mathbb{A}^2_{\mathbb{C},X}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}})$ and $t \in \Gamma(\mathbb{A}^2_{\mathbb{C},Y}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}})$, since $\Gamma(\mathbb{A}^2_{\mathbb{C},X}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) = k[X,Y]_X$, $\Gamma(\mathbb{A}^2_{\mathbb{C},Y}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) = k[X,Y]_Y$ there exist two polynomial $f, g \in k[X,Y]$ such that $X \nmid f$, $Y \nmid g$ and $s = \frac{f}{X^m}$, $g = \frac{t}{Y^n}$ for some $m, n \geqslant 0$. On the other hand $\Gamma(\mathbb{A}^2_{\mathbb{C},XY}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) = k[X,Y]_{XY}$, thus $s_{|_{\mathbb{A}^2_{\mathbb{C},XY}}} - t_{|_{\mathbb{A}^2_{\mathbb{C},XY}}} = 0$ if and only if

$$f \cdot Y^n = g \cdot X^m,$$

   and this is possible if and only if $n = m = 0$ and $f = g$. Hence $\Gamma(\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) = k[X,Y]$.

   (b) By contradiction, assume $\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}$ affine scheme, then

$$\mathbb{A}^2_{\mathbb{C}} \setminus \{0\} = \text{Spec}(\Gamma(\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}})) = \text{Spec}(k[X,Y]) = \mathbb{A}^2_{\mathbb{C}},$$

   and this is absurd.

4. In the following if X is a scheme we denote by $\mathrm{sp}(X)$ the underlying topological space of X. Let S be a scheme and $\pi : X \to S$, $\rho : Y \to S$ be S-schemes. Let $\mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$ be the fiber product of sets defined by π and ρ, endowed with the topology induced by the product topology on $\mathrm{sp}(X) \times \mathrm{sp}(Y)$. We are going to study some property concerning the relation between $\mathrm{sp}(X \times Y)$ and $\mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$.

   (a) Show that we have a canonical map $f : \mathrm{sp}(X \times_S Y) \to \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$.

   (b) Show that f is surjective.

   (c) Let us consider the example $X = Y = \mathrm{Spec}\,\mathbb{C}$ and $S = \mathrm{Spec}\,\mathbb{R}$. Show that $X \times_S Y \cong \mathrm{Spec}(\mathbb{C} \oplus \mathbb{C})$ and that f is not injective.

   (d) Show that in the case of the previous Exercise, with $X = \mathrm{Spec}\,k(u)$, $Y = \mathrm{Spec}\,k(v)$ and $S = \mathrm{Spec}\,k$, the map f has infinite fibers.

   (e) Let $S = \mathrm{Spec}\,k$ be the spectrum of an arbitrary field. By studying the example $X = Y = \mathbb{A}^1_k$, show that the image of an open subset under f is not necessarily an open subset.

Solution:

   (a) By definition of the fiber product of scheme we have two morphism $\lambda_1 : X \times_S Y \to X$ and $\lambda_2 : X \times_S Y \to Y$ such that diagram

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\lambda_1} & X \\ \downarrow{\scriptstyle \lambda_2} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\rho} & S \end{array}$$

   is commutative. Then we define

$$\begin{array}{rcl} f : \mathrm{sp}(X \times Y) & \to & \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y) \\ z & \mapsto & (\lambda_1(z), \lambda_2(z)). \end{array}$$

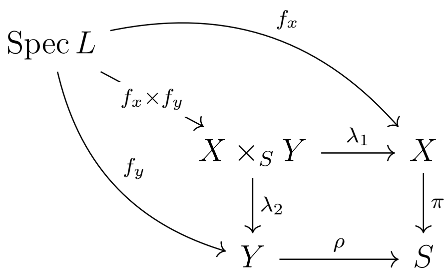
   (b) Let $(x, y) \in \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$ we want to show that there exist $z \in \mathrm{sp}(X \times_S Y)$ such that $f(z) = (x, y)$. Let us denote by $k(x), k(y)$ the residue field of x, y respectively. Since $(x, y) \in \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$ we have that $\pi(x) = \rho(y) = s \in S$. Moreover we also get $k(s) \hookrightarrow k(x)$ and $k(s) \hookrightarrow k(y)$ since $\pi : X \to S$ and $\rho : Y \to S$ are morphisms. Since $k(x), k(y) \supset k(s)$ we get that $k(x) \otimes_{k(s)} k(y) = k(x).k(y)$ the compositum field of $k(x)$ and $k(y)$. We denote $L := k(x)k(y)$ the compositum of $k(x)$ and $k(y)$. So we get the two morphisms

$$f_x : \mathrm{Spec}\,L \to X \qquad f_y : \mathrm{Spec}\,L \to Y,$$

such that $f_x((0)) = x$ and $f_y((0)) = y$. Thus we have the commutative diagram of scheme

$$\begin{array}{ccc} \operatorname{Spec} L & \xrightarrow{f_x} & X \\ \big\downarrow{\scriptstyle f_y} & & \big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\rho} & S. \end{array}$$

Using the the universal property of $X \times_S Y$ there exists a morphism of scheme $f_x \times f_y : \operatorname{Spec} L \to X \times_S Y$ such that



is a commutative diagram. Then $z := f_x \times f_y((0))$ is such that $f(z) = (x, y)$, indeed

$$\lambda_1(z) = \lambda_1 \circ f_x \times f_y((0)) = f_x((0)) = x$$

and similarly for λ_2 and y.

(c) Let $X = Y = \operatorname{Spec} \mathbb{C}$ and $S = \operatorname{Spec} \mathbb{R}$. To show that $X \times_S Y \cong \operatorname{Spec}(\mathbb{C} \oplus \mathbb{C})$ it is enough to show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$. An explicit isomorphism is given for example by

$$\begin{array}{ccc} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} & \to & \mathbb{C} \oplus \mathbb{C} \\ z \otimes w & \mapsto & (z \cdot w, z \cdot \overline{w}). \end{array}$$

Then f is not injective since $|\operatorname{sp}(\operatorname{Spec} \mathbb{C}) \times_{\operatorname{sp}(\operatorname{Spec} \mathbb{R})} \operatorname{sp}(\operatorname{Spec} \mathbb{C})| = 1$ while $|\operatorname{sp}(\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C})| = 2$.

(d) We have that $\operatorname{sp}(\operatorname{Spec}(k(u))) \times_{\operatorname{sp}(\operatorname{Spec} k)} \operatorname{sp}(\operatorname{Spec}(k(v))) = \{((0), (0)\}$, thus $f^{-1}((0), (0)) = \operatorname{Spec} A$. On the other hand $|\operatorname{Spec} A| = \infty$ thanks to part (d) of the previous exercise.

(e) Since f is surjective it is enough to show that the image of a closed subset under f is not necessarily closed. Let $X = Y = \mathbb{A}^1_k$, then we know that $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$ with respect to the Zarisky topoly. On the other hand the topology in $\operatorname{sp}(\mathbb{A}^1_k) \times_{\operatorname{sp}(\operatorname{Spec} k)} \operatorname{sp}(\mathbb{A}^1_k)$ is the product topology, i.e. $Z \subset \operatorname{sp}(\mathbb{A}^1_k) \times_{\operatorname{sp}(\operatorname{Spec} k)} \operatorname{sp}(\mathbb{A}^1_k)$ is closed if and only if $Z = V(I) \times_{\operatorname{sp}(\operatorname{Spec} k)} V(J)$, where $I, J \subset k[X]$ are ideals. Consider $g = X + Y$. Then we have that

$$f(V((g))) = \{((X + a)(X - a)) : a \in k\} \cup \{((0)(0))\}$$

which is not of the for $V(I) \times_{\operatorname{sp}(\operatorname{Spec} k)} V(J)$. Thus $g(V((g))$ is not closed.

5. Let $\mathcal{C}$ be a category and $X \in \mathrm{Ob}(\mathcal{C})$, one defines

$$h_X : \begin{array}{ccc} \mathcal{C} & \to & (\mathrm{Sets}) \\ Y & \mapsto & \mathrm{Hom}(Y, X) \end{array}.$$

   (a) Show that h_X is a controvariant functor.

   (b) Show that any morphism $f : X_1 \to X_2$ induces a morphism of functors

$$h_f : h_{X_1} \to h_{X_2}.$$

   (c) Conversely, let $\varphi : h_{X_1} \to h_{X_2}$ be a morphism of functors. There is an unique $f : X_1 \to X_2$ such that $\varphi = h_f$.
   Solution: This result is called *Yoneda Lemma*. You can find a proof of this in Szamuely's book "*Galois Groups and Fundamental Groups*", pp. $20, 21$.

6. Let S be a scheme and consider the category $\mathcal{C} =$ (Schemes over S)

   (a) If $X \to S$ is a scheme over S such that

$$h_X : T \to \mathrm{Hom}_S(T, X) = X(T)$$

   is a functor to groups then X has a structure of S-group scheme.

   (b) Consider $\mathbb{G}_m := \mathrm{Spec}(\mathbb{Z}[X, X^{-1}])$. Prove that

$$\mathbb{G}_m(T) = \mathcal{O}_T(T)^\times$$

   for any scheme T and conclude that $\mathbb{G}_m$ is a group scheme. Moreover, describe the morphism of rings corresponding to the multiplication

$$m : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m.$$

Solution:

   (a) We need to check the axioms of S-group scheme, i.e. we need tgo define three maps

$$m : X \times_S X \to X, \quad e : S \to X, \quad i : X \to X,$$

   subject to the commutative diagrams

$$\begin{array}{ccc} X \times_S X \times_S X & \xrightarrow{\mathrm{Id} \times m} & X \times_S X \\ \downarrow{\scriptstyle m \times \mathrm{Id}} & & \downarrow{\scriptstyle m} \\ X \times_S X & \xrightarrow{\quad m \quad} & X. \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\mathrm{Id} \times e} & X \times_S X \\ \downarrow{\scriptstyle e \times \mathrm{Id}} & \searrow^{\mathrm{Id}} & \downarrow{\scriptstyle m} \\ X \times_S X & \xrightarrow{\quad m \quad} & X. \end{array}$$

and

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{Id}\times i} & X\times_S X \\ \big\downarrow{\scriptstyle i\times \mathrm{Id}} & \searrow^{e\circ p} & \big\downarrow{\scriptstyle m} \\ X\times_S X & \xrightarrow{m} & X, \end{array}$$

where $p : X \to S$ is the structure morphism. We start defining the map m. For any S-scheme T, we know that $X(T)$ is a group. Thus, we consider the multiplication map, $\mu(T)$, over $X(T)$:

$$\mu(T) : X(T) \times X(T) \to X(T).$$

On the other hand, for any S-scheme T one has that $X(T) \times X(T) = (X \times_S X)(T)$ (Remark 1.6 page 81 in Liu's book "*Algebraic Geometry and Arithmetic Curves*"), thus for any S-scheme T we have a map

$$\mu(T) : (X \times_S X)(T) \to X(T).$$

Let us check that these maps define a functor $\mu : h_{X\times X} \to h_X$. Let T, T' be S-schemes and $t : T \to T'$ a S-morphism, then we have the diagram

$$\begin{array}{ccc} (X\times_S X)(T') = X(T')\times X(T') & \xrightarrow{\mu(T')} & X(T') \\ \big\downarrow{\scriptstyle h_{X\times X}(t)} & & \big\downarrow{\scriptstyle h_X(t)} \\ (X\times_S X)(T) = X(T)\times X(T) & \xrightarrow{\mu(T)} & X(T), \end{array}$$

which is commutative since h_X is a functor to groups. Thanks to part (c) of the previous exercise, there exists an unique $m : X \times_S X \to X$ such that $h_m = \mu$. note that one can also define m using the fiber product:

$$\begin{array}{ccc} X\times_S X & \xrightarrow{m} & X \\ \big\downarrow{\scriptstyle m} & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & S. \end{array}$$

To define i, one argue as before using the fact that for any S-scheme T, we have an inversion map

$$\iota(T) : X(T) \to X(T).$$

Let us conclude defining e. One can see $\mathrm{id}_S : S \to S$ as an S-scheme. Then we define e as the zero element in $\mathrm{Hom}_S(S, X)$. To verify that the maps m, i, e satisfy the commutative diagrams is just a computation involving part $(b), (c)$ of the previous exercise and the universal property of the fiber product.

(b) We have that

$$\mathbb{G}_m(T) = \mathrm{Hom}(T, \mathrm{Spec}(\mathbb{Z}[X, X^{-1}])) = \mathrm{Hom}(\mathbb{Z}[X, X^{-1}], \mathcal{O}_T(T)) = \mathcal{O}_T(T)^\times,$$

where the second step follows from the fact that $\mathbb{G}_m$ is an affine scheme. Hence, $\mathbb{G}_m(T)$ is a group for any scheme T. Let us check that $h_{\mathbb{G}_m}$ sends morphisms to group homomorphism: let T, T' be schemes and $t : T \to T'$ a morphism, and denote by $t^\sharp : \mathcal{O}_{T'}(T') \to \mathcal{O}_T(T)$. Then we have the map

$$h_{\mathbb{G}_m}(t) : \begin{array}{ccc} \mathbb{G}_m(T') & \to & \mathbb{G}_m(T) \\ \psi & \mapsto & \psi \circ t. \end{array}$$

Using again the fact that $\mathbb{G}_m(T') = \mathrm{Hom}(\mathbb{Z}[X, X^{-1}], \mathcal{O}_{T'}(T'))$ and that $\mathbb{G}_m(T) = \mathrm{Hom}(\mathbb{Z}[X, X^{-1}], \mathcal{O}_T(T))$, we can rewrite $h_{\mathbb{G}_m}(t)$ as

$$h_{\mathbb{G}_m}(t) : \begin{array}{ccc} \mathrm{Hom}(\mathbb{Z}[X, X^{-1}], \mathcal{O}_{T'}(T')) & \to & \mathrm{Hom}(\mathbb{Z}[X, X^{-1}], \mathcal{O}_T(T)) \\ X \mapsto x & \mapsto & X \mapsto t^\sharp(x), \end{array}$$

i.e. $h_{\mathbb{G}_m}(t) = t^\sharp_{|_{\mathcal{O}^\times_{T'}(T')}} : \mathcal{O}^\times_{T'}(T') \to \mathcal{O}^\times_T(T)$. Thus, $h_{\mathbb{G}_m}(t)$ is a group homomorpism and we can conclude that $\mathbb{G}_m$ is a group scheme thanks to part (a). Let us describe m. First of all, observe that

$$\mathbb{G}_m \times \mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[X_1, X_1^{-1}] \otimes \mathbb{Z}[X_2, X_2^{-1}]).$$

Thus, we need to describe

$$m^\sharp : \mathbb{Z}[X, X^{-1}] \to \mathbb{Z}[X_1, X_1^{-1}] \otimes \mathbb{Z}[X_2, X_2^{-1}].$$

Let us consider the commutative diagram

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\mathrm{Id}\times e} & \mathbb{G}_m \times \mathbb{G}_m \\ \downarrow{\scriptstyle e\times\mathrm{Id}} & \searrow^{\mathrm{Id}} & \downarrow{\scriptstyle m} \\ \mathbb{G}_m \times \mathbb{G}_m & \xrightarrow{m} & \mathbb{G}_m \end{array} .$$

From this diagram we get the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[X, X^{-1}] & \xrightarrow{m^\sharp} & \mathbb{Z}[X_1, X_1^{-1}] \otimes \mathbb{Z}[X_2, X_2^{-1}] \\ \downarrow{\scriptstyle m^\sharp} & \searrow^{\mathrm{Id}} & \downarrow{\scriptstyle \mathrm{Id}_1\otimes e^\sharp} \\ \mathbb{Z}[X_1, X_1^{-1}] \otimes \mathbb{Z}[X_2, X_2^{-1}] & \xrightarrow{\mathrm{Id}_2\otimes e^\sharp} & \mathbb{Z}[X, X^{-1}] \end{array} ,$$

where for any $a, b \in \mathbb{Z}$ one has $\mathrm{Id}_i(aX_i) = aX$, $e^\sharp(bX_i) = b$ for $i = 1, 2$. Thus, $(\mathrm{Id}_1 \otimes e^\sharp)(m^{\sharp(X)}) = (\mathrm{Id}_2 \otimes e^\sharp)(m^\sharp(X)) = X$. On the other hand, we have that

$$m^\sharp(X) = \sum_{i,j=-1}^{1} a_{i,j}(X_1^i \otimes X_2^j)$$

thus we get $a_{1,0} + a_{0,0} + a_{-1,0} = 0 = a_{0,1} + a_{0,0} + a_{0,-1}$, $a_{1,0} + a_{1,1} + a_{1,-1} = 1 = a_{0,1} + a_{1,1} + a_{-1,1}$ and $a_{-1,-1} + a_{-1,0} + a_{-1,1} = 0 = a_{-1,-1} + a_{0,-1} + a_{1,-1}$. Looking at the other diagram

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\mathrm{Id}\times i} & \mathbb{G}_m \times \mathbb{G}_m \\ \Big\downarrow{\scriptstyle i\times \mathrm{Id}} & \overset{e\circ p}{\searrow} & \Big\downarrow{\scriptstyle m} \\ \mathbb{G}_m \times \mathbb{G}_m & \xrightarrow{m} & \mathbb{G}_m, \end{array}$$

one gets the relations $a_{1,-1} = a_{-1,1} = 0$, $a_{0,0} + a_{1,1} + a_{-1,-1} = 1$, $a_{1,0} + a_{0,-1} = a_{-1,0} + a_{0,1} = 0$. Putting all the conditions together we get that $a_{1,1} = 0$ and $a_{i,j} = 0$ for $(i, j) \neq (1, 1)$. Thus $m^\sharp(X) = X_1 \otimes X_2$ and this concludes the exercise.