REPRESENTATION THEORY OF LIE GROUPS EXERCISES

Exercise 1. *Let G be a finite group. Deduce from the Peter–Weyl theorem that*

$$\#G = \sum (\dim V)^2,$$

with the sum taken over the isomorphism classes of finite-dimensional irreducible representations of G.

Solution 1. Peter–Weyl says that the characters of the irreducible representations π give an orthonormal basis for the space of class functions f, thus $\|f\|^2 = \sum_\pi |\langle f, \chi_\pi \rangle|^2$. Taking $f(g) = 1$ if $g = e$ and 0 otherwise gives $\|f\|^2 = 1/\#G$ and $\langle f, \chi_\pi \rangle = \chi_\pi(1)/\#G = \dim(\pi)/\#G$. Summing over π gives what we want.

Alternatively, use the decomposition of the coefficient ring $\mathcal{A}(G)$ as the direct sum of $\mathcal{A}(\pi)$ over irreducibles π, use that $\mathcal{A}(\pi) \cong \mathrm{End}(\pi)^*$, and compare dimensions: $\dim \mathcal{A}(G) = \#G$, while $\dim \mathrm{End}(\pi)^* = (\dim \pi)^2$.

Exercise 2. *Let G_1, G_2 be compact groups. For $j = 1, 2$, let π_j be a finite-dimensional representation of G_j. The external tensor product $\pi_1 \boxtimes \pi_2$ is the representation of $G_1 \times G_2$ defined by $(g_1, g_2)(v_1 \otimes v_2) := g_1 v_1 \otimes g_2 v_2$. Compute $\chi_{\pi_1 \boxtimes \pi_2}$ in terms of χ_{π_1} and χ_{π_2}. Show that the map*

$$(\pi_1, \pi_2) \mapsto \pi_1 \boxtimes \pi_2$$

induces a bijection between $\mathrm{Irr}(G_1) \times \mathrm{Irr}(G_2)$ and $\mathrm{Irr}(G_1 \times G_2)$; here, as usual, we denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of irreducible finite-dimensional representations of a compact group G.

Solution 2. Let $\{u_i\}_i$ and $\{v_j\}$ be two ONBs of π_1 and π_2, respectively. So that, $\{u_i \otimes v_j\}_{i,j}$ is an ONB of $\pi_1 \boxtimes \pi_2$. Hence,

$$\begin{aligned}
\chi_{\pi_1 \boxtimes \pi_2}(g_1, g_2) &= \sum_{i,j} \langle \pi_1 \boxtimes \pi_2(g_1, g_2) u_i \otimes v_j, u_i \otimes v_j \rangle_{\pi_1 \boxtimes \pi_2} \\
&= \sum_{i,j} \langle \pi_1(g_1) u_i, u_i \rangle_{\pi_1} \langle \pi_2(g_2) v_j, v_j \rangle_{\pi_2} \\
&= \chi_{\pi_1}(g_1) \chi_{\pi_2}(g_2).
\end{aligned}$$

First we show that the map is well-defined, that is, $\pi_1 \boxtimes \pi_2$ is irreducible, if π_1 and π_2 are. To check that we use the irreducibly criteria of the character that it is a unit in L^2. Indeed,

$$\|\chi_{\pi_1 \boxtimes \pi_2}\|_{L^2(G_1 \times G_2)} = \|\chi_{\pi_1}\|_{L^2(G_1)} \|\chi_{\pi_2}\|_{L^2(G_2)} = 1.$$

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Next, to check injectivity, let $\pi_1 \boxtimes \pi_2 \cong \pi_1' \boxtimes \pi_2'$, so that

$$\chi_{\pi_1}(g_1)\chi_{\pi_2}(g_2) = \chi_{\pi_1'}(g_1)\chi_{\pi_2'}(g_2), \quad \forall g_i \in G_i.$$

Choosing $g_2 = 1$ we get $\chi_{\pi_1} = c\chi_{\pi_1'}$. From irreducibility of π_i one gets that $c = 1$, hence $\pi_1 \cong \pi_2$. Similarly, $\pi_2 \cong \pi_2'$.

Finally, to check surjectivity we invoke Peter-Weyl. Let π be an irreducible representation of $G_1 \times G_2$ which is not isomorphic to any $\pi_1 \boxtimes \pi_2$ for $\pi_i \in \mathrm{Irr}(G_i)$. So

$$\chi_\pi \perp \chi_{\pi_1 \boxtimes \pi_2}, \quad \forall \pi_i \in \mathrm{Irr}(G_i).$$

But $\{\chi_{\pi_i}\}_{\pi_i \in \mathrm{Irr}(G_i)}$ is an ONB of $L^2(G_i)$, so that, $\{\chi_{\pi_1}\chi_{\pi_2}\}$ is an ONB of $L^2(G_1 \times G_2)$. But then, $\chi_\pi \perp L^2(G_1 \times G_2)$, which is a contradiction.

Exercise 3. *For $n \geq 1$, let V be the regular representation of $\mathbb{Z}/n\mathbb{Z}$, thus V consists of functions $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$. Define a linear map $T : V \to V$ by*

$$Tf(x) := \frac{f(x-1) + f(x+1)}{2}.$$

Define $f_0 \in V$ by

$$f_0(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define f_k inductively by $f_{k+1} := Tf_k$.

1. *Verify that T is equivariant.*
2. *Decompose V as a sum of inequivalent irreducible invariant subspaces W, and compute the eigenvalue of T on each W.*
3. *Suppose that n is odd. Show that there exist positive constants C, c (not depending upon n or k) so that*

$$\sum_{x \in \mathbb{Z}/n\mathbb{Z}} |f_k(x) - 1/n|^2 \leq Cn^2 \exp(-ck/n^2).$$

   Thus, informally, f_k is rather uniform for k a bit larger than n^2. [Hint: write the LHS in terms of the eigenvalues of T on the nontrivial W.]
4. *What happens if n is even?*

Solution 3. (1) We compute

$$\begin{aligned} T \circ \pi(y) f(x) &= \frac{\pi(y)f(x-1) + \pi(y)f(x+1)}{2} \\ &= \frac{f(x+y-1) + f(x+y-1)}{2} \\ &= Tf(x+y) = \pi(y) \circ Tf(x). \end{aligned}$$

Hence T is equivariant.

(2) As the group is abelian the irreducible representations are one dimensional. We claim that $\{e_k\}_{k\in\mathbb{Z}/n\mathbb{Z}}$ are all possible characters, where $e_k(x) := e^{2\pi ixk/n}$. Consequently,

$$V = \oplus_{k\in\mathbb{Z}/n\mathbb{Z}}\mathbb{C}e_k.$$

The claim follows from the fact that the characters are inequivalent and by the Fourier inversion formula any $f \in L^2(\mathbb{Z}/n\mathbb{Z})$ can be written as

$$f(x) = \sum_{l\in\mathbb{Z}/n\mathbb{Z}} \hat{f}(l)e_l(x),$$

where the Fourier transform is as defined in the lecture.

Finally, we compute the T-eigenvalues:

$$Te_l(x) = \frac{e_l(x-1) + e_l(x+1)}{2} = e_l(x)\cos(2l\pi/n).$$

(3) Let us start by writing f_0 in its Fourier expansion as

$$f_0(x) = \frac{1}{n}\sum_{x\in\mathbb{Z}/n\mathbb{Z}} \hat{f}(l)e_l(x) = \frac{1}{n}\sum_{l\in\mathbb{Z}/n\mathbb{Z}} e_l(x).$$

Thus from (2),

$$f_k(x) = T^k f_0(x) = \frac{1}{n}\sum_{l\in\mathbb{Z}/n\mathbb{Z}} \cos^k(2\pi l/n)e_l(x).$$

Hence using that for n odd

$$|\cos(2\pi l/n)| \leq 1 - c/n^2 \leq \exp(-c/n^2),$$

for some positive constant c. Thus

$$\begin{aligned}\sum_{x\in\mathbb{Z}/n\mathbb{Z}} |f_k(x) - 1/n|^2 &= \frac{1}{n^2}\sum_{x\in\mathbb{Z}/n\mathbb{Z}}\left|\sum_{0\neq l\in\mathbb{Z}/n\mathbb{Z}} \cos^k(2\pi l/n)e_l(x)\right|^2 \\ &\leq \frac{n(n-1)^2)}{n^2}\exp(-ck/n^2),\end{aligned}$$

by Cauchy-Schwarz and trivially bounding all the characters.

Exercise 4. *For the representations* $\mathrm{Sym}^k(\mathbb{C}^n)$ *and* $\bigwedge^k(\mathbb{C}^n)$ *of* $\mathrm{U}(n)$*, describe the weights, determine the lexicographically highest weight, and use the Weyl character formula to prove that these representations are irreducible.*

Solution 4. We work out the $\bigwedge^k(\mathbb{C}^n)$ case, $\mathrm{Sym}^k(\mathbb{C}^n)$ would be similar (also done as a sketch in the notes). We recall that

$$\chi_{\wedge^k(\mathbb{C}^n)}(t) = \sum_{i_1<\cdots<i_k} t_{i_1}\ldots t_{i_k}.$$

Thus the highest weight is $\lambda := (1, \ldots, 1)$. Thus the dimension of the irreducible subrepresentation V_λ attached to the highest weight, by the Weyl dimension formula, is

$$\prod_{i=1}^{k} \prod_{j=k+1}^{n} \frac{1+j-i}{j-i} = \binom{n}{k} = \dim(\bigwedge^k(\mathbb{C}^n)).$$

Thus $\bigwedge^k(\mathbb{C}^n) = V_\lambda$ hence the claim follows.

Exercise 5. *Use the Weyl character formula to describe the decomposition into irreducibles of the representations* $\mathrm{Sym}^{k_1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{k_2}(\mathbb{C}^2)$ *of* $\mathrm{U}(2)$*.*

Solution 5. We know the character of $\mathrm{Sym}^k(\mathbb{C}^2)$ is given by

$$\chi_k(\mathrm{diag}(t_1, t_2)) = \sum_{i=0}^{k} t_1^i t_2^{k-i} = \frac{t_1^{k+1} - t_2^{k+1}}{t_1 - t_2}.$$

Let WOLG $k_1 \leq k_2$. Hence for characters χ_{k_i} of $\mathrm{Sym}^{k_i}(\mathbb{C}^2)$ we get

$$\begin{aligned}
\chi_{k_1}(t)\chi_{k_2}(t) &= \frac{t_1^{k_2+1} - t_2^{k_2+1}}{t_1 - t_2} \sum_{i=0}^{k_1} t_1^i t_2^{k_1-i} \\
&= \sum_{i=0}^{k_1} \frac{t_1^{k_2+1+i} t_2^{k_1-i} - t_1^{k_1-i} t_2^{k_2+1+i}}{t_1 - t_2} = \sum_{i=0}^{k_1} (t_1 t_2)^{k_1-i} \frac{t_1^{k_2+1-k_1+2i} - t_2^{k_2+1-k_1+2i}}{t_1 - t_2} \\
&= \chi_{k_1+k_2}(t) + t_1 t_2 \chi_{k_1+k_2-2}(t) + (t_1 t_2)^2 \chi_{k_1+k_2-4}(t) \ldots.
\end{aligned}$$

Thus

$$\mathrm{Sym}^{k_1}(\mathbb{C}^2) \otimes \mathrm{Sym}^{k_2}(\mathbb{C}^2) = \bigoplus_{l=0}^{\min(k_1,k_2)} (\det)^i \otimes \mathrm{Sym}^{k_1+k_2-2i}(\mathbb{C}^2).$$

It is easy to check (by Irreducibility criteria) that the representations in the summands are irreducible.

Exercise 6. *For* $0 \leq k \leq n$*, establish the existence of the following isomorphism of representations of* $\mathrm{U}(n)$*:*

$$\bigwedge^k(\mathbb{C}^n) \cong \bigwedge^{n-k}(\mathbb{C}^n)^* \otimes \det.$$

Solution 6. We check by the equality of the characters in the both sides.

$$\chi_{\wedge^k(\mathbb{C}^n)}(t) = \sum_{i_1 < \cdots < i_k} t_{i_1} \ldots t_{i_k},$$

and

$$\chi_{\wedge^{n-k}(\mathbb{C}^n)^*}(t) = \sum_{i_1 < \cdots < i_{n-k}} t_{i_1}^{-1} \ldots t_{i_{n-k}}^{-1}.$$

The claim follows immediately from $\det(g) = \prod_i t_i$.

Exercise 7. *Recall that* $\mathrm{SU}(n) = \ker(\det : G \to \mathrm{U}(1))$. *Let* $Z \cong \mathrm{U}(1)$ *denote the center of* $\mathrm{U}(n)$, *and set* $\mathrm{PU}(n) := \mathrm{U}(n)/Z$. *Let* $\mu_n \subseteq Z$ *denote the subgroup of* n*th roots of unity, so that* $\mu_n = Z \cap \mathrm{SU}(n)$. *Observe that*

- *given a representation of* $\mathrm{PU}(n)$, *we obtain a representation of* $\mathrm{U}(n)$ *by pullback, i.e., by composing with the projection* $\mathrm{U}(n) \twoheadrightarrow \mathrm{PU}(n)$, *and*
- *given a representation of* $\mathrm{U}(n)$, *we obtain a representation of* $\mathrm{SU}(n)$ *by restriction, i.e., by composing with the inclusion* $\mathrm{SU}(n) \to \mathrm{U}(n)$.

Show that the operations of pullback and restriction just described preserve irreducibility, inducing an injective map

$$\mathrm{Irr}(\mathrm{PU}(n)) \hookrightarrow \mathrm{Irr}(\mathrm{U}(n))$$

and a surjective map

$$\mathrm{Irr}(\mathrm{U}(n)) \twoheadrightarrow \mathrm{Irr}(\mathrm{SU}(n)).$$

Show that the latter maps and the bijection $\mathrm{Irr}(\mathrm{U}(n)) \leftrightarrow \{$*dominant elements of* $\mathbb{Z}^n\}$ *as in the Weyl character formula are compatible with bijections*

$$\mathrm{Irr}(\mathrm{SU}(n)) \leftrightarrow \{\textit{dominant elements of } \mathbb{Z}^n/\mathbb{Z}(1,\dots,1)\}$$

and

$$\mathrm{Irr}(\mathrm{PU}(n)) \leftrightarrow \{\textit{dominant elements of } (\mathbb{Z}^n)_0 := \{\lambda \in \mathbb{Z}^n : \sum_j \lambda_j = 0\}\}.$$

Solution 7. Let $\pi_i \in \mathrm{Irr}(\mathrm{PU}(n))$ for $i = 1, 2$ such that $\tilde{\pi}_1 \cong \tilde{\pi}_2$ where $\tilde{\pi}_i$ is the representation of $\mathrm{U}(n)$ obtained by pulling back. Thus, $\chi_{\tilde{\pi}_1} = \chi_{\tilde{\pi}_2}$ on $\mathrm{U}(n)$, hence equal on $\mathrm{PU}(n)$. Consequently, $\pi_1 \cong \pi_2$. If π is irreducible then $\|\chi_\pi\|_{L^2(\mathrm{PU}(n))} = 1$. Note that, by definition of the pull back map the center Z acts trivially on $\tilde{\pi}$. Thus $\chi_{\tilde{\pi}}(zg) = \chi_{\tilde{\pi}}(g)$. We calculate under a probability Haar measure of $\mathrm{U}(n)$

$$\int_{\mathrm{U}(n)} |\chi_{\tilde{\pi}}(g)|^2 = \int_Z \int_{\mathrm{PU}(n)} |\chi_{\tilde{\pi}}(zg)|^2 = \frac{1}{\mathrm{vol}(\mathrm{PU}(n))} \int_{\mathrm{PU}(n)} |\chi_\pi(g)|^2 \frac{1}{\mathrm{vol}(Z)} \int_Z 1 = 1.$$

Thus $\tilde{\pi}$ is irreducible.

Let π be a representation of $\mathrm{U}(n)$ with $\bar{\pi}$ corresponding restriction representation of $\mathrm{SU}(n)$. We check irreducibility in a similar way as previous. Let π be irreducible, so that $\|\chi_\pi\|_{L^2(\mathrm{U}(n))} = 1$. Note that, for $\alpha \in \mathrm{U}(1) = \mathrm{U}(n)/\mathrm{SU}(n)$ we have $\chi_\pi(\alpha g) = \chi(\det(\alpha))\chi_\pi(g)$, for some unitary character χ of $\mathrm{U}(1)$. Obviously, $\chi_{\bar{\pi}}(g) = \chi_\pi(g)$ for $g \in \mathrm{SU}(n)$. Thus under a probability Haar measure of $\mathrm{U}(n)$

$$\int_{\mathrm{U}(n)} |\chi_\pi(g)|^2 = \int_{\mathrm{U}(1)} \int_{\mathrm{SU}(n)} |\chi_\pi(\alpha g)|^2 = \int_{\mathrm{SU}(n)} |\chi_{\bar{\pi}}(g)|^2 \int_{\mathrm{U}(1)} |\chi(\det(\alpha))|^2 = \frac{1}{\mathrm{vol}(\mathrm{SU}(n))} \int_{\mathrm{SU}(n)} |\chi_{\bar{\pi}}(g)|^2.$$

Thus $\bar{\pi}$ is irreducible. To check surjectivity let $\pi \in \mathrm{Irr}(\mathrm{SU}(n))$. We define $\tilde{\pi} \in \mathrm{Irr}(\mathrm{U}(n))$ by

$$\tilde{\pi}(g) = \det(g)\pi(g/(\det(g)^{1/n}).$$

Clearly, $\bar{\tilde{\pi}} = \pi$. Irreducibility follows from the previous computation.

Let $\pi \in \mathrm{Irr}(\mathrm{SU}(n))$. We construct $\tilde{\pi} \in \mathrm{Irr}(\mathrm{U}(n))$, as previous, so that $\tilde{\pi}$ restricts on $\mathrm{SU}(n)$ as π. Let $\lambda \in \mathbb{Z}^n$ is such that $\chi_{\tilde{\pi}} = s_\lambda$. For $t \in T \cap \mathrm{SU}(n)$, so that $\det(t) = 1$, we get $\forall k \in \mathbb{Z}$

$$\chi_\pi(t) = s_\lambda(t) = s_{\lambda-(k,\dots,k)}.$$

Thus $\pi|_{\mathrm{SU}(n)}$ corresponds to $\lambda + \mathbb{Z}(1,\dots,1)$, hence the correspondence is compatible.

Similarly, if $\pi \in \mathrm{Irr}(\mathrm{PU}(n))$, we pull back π to $\tilde{\pi} \in \mathrm{Irr}(\mathrm{U}(n))$. Let $\lambda \in \mathbb{Z}^n$ so that $\chi_{\tilde{\pi}} = s_\lambda$. As $\forall z \in Z$ we have $\chi_{\tilde{\pi}}(zg) = \chi_{\tilde{\pi}}(g)$, we obtain

$$z^{\sum_i \lambda_i} s_\lambda(t) = s_\lambda(zt) = s_\lambda(t), \quad \forall t \in T.$$

Thus $\sum \lambda_i = 0$, i.e., the correspondence is compatible.

Exercise 8. *(Knapp, Exercise 11.4) Fix $n \geq 2$. For nonnegative integers p, q, let $V_{p,q}$ denote the space of polynomials $f \in \mathbb{C}[z_1, \dots, z_n, \overline{z_1}, \dots, \overline{z_n}]$ that are bihomogeneous of degree (p,q) with respect to the z_i's and the $\overline{z_j}$'s, that is to say,*

$$V_{p,q} := \oplus_{\substack{i_1 \leq \cdots \leq i_p \\ j_1 \leq \cdots \leq j_q}} \mathbb{C} z_{i_1} \cdots z_{i_p} \overline{z_{j_1} \cdots z_{j_q}}.$$

We may identify $V_{p,q}$ with a space of functions $f : \mathbb{C}^n \to \mathbb{C}$ given by polynomials in the real and imaginary parts of the argument. Regarding $\mathbb{C}^n$ as the space of row vectors $(z_1, \dots, z_n)$, the group $\mathrm{SU}(n)$ acts on $V_{p,q}$ by the rule $g \cdot f(z) := f(zg)$.

(1) *Verify that $V_{p,q}$ actually defines a representation of $\mathrm{SU}(n)$.*
(2) *Write down a natural isomorphism*

$$V_{p,q} \cong \mathrm{Sym}^p(\mathbb{C}^n) \otimes \mathrm{Sym}^q((\mathbb{C}^n)^*). \tag{0.1}$$

Determine the weights of $V_{p,q}$.
(3) *Show that the Laplacian $\Delta := \sum_j \frac{\partial}{\partial z_j} \frac{\partial}{\partial \overline{z_j}}$ defines a equivariant surjection $\Delta : V_{p,q} \to V_{p-1,q-1}$.*
(4) *Let $H_{p,q} \leq V_{p,q}$ denote the space of harmonic polynomials, i.e., the kernel of Δ. Show that $H_{p,q}$ is an invariant subspace. Compute its dimension and lexicographically highest weight. Show that it is irreducible.*
(5) *Show that every irreducible representation of $\mathrm{SU}(n)$ is isomorphic to some $H_{p,q}$ if and only if $n \leq 3$.*

Solution 8. (1) This is clear if we check that if f is a monomial of degree (p,q) then $f(zg)$ is a bihomogeneous polynomial of degree (p,q). Let $f = \prod_{i=1}^n z_i^{s_i} \bar{z}_i^{t_i}$ with $\sum_i s_i = p$ and $\sum t_i = q$. Then

$$f(zg) = \prod_{i=1}^n (\sum_j g_{ij} z_j)^{s_i} (\overline{\sum_j g_{ij} z_j})^{t_i}.$$

Each factor is bihomogeneous of degree (k,l), so is $f(zg)$.

(2) Let $\{e_i\}$ and $\{e_i^*\}$ be the standard basis of $\mathbb{C}^n$ and $\mathbb{C}^{*n}$. The idea is to "map z_i to e_i". Choose the standard basis in $V_{p,q}$ to be

$$\{\prod_{i=1}^n z_i^{s_i}\bar{z}_i^{t_i} \mid \sum_i s_i = p, \sum t_i = q\}.$$

We claim that

$$\prod_{i=1}^n z_i^{s_i}\bar{z}_i^{t_i} \mapsto \otimes_{i=1}^n e_i^{\otimes s_i} e_i^{*\otimes t_i}$$

is $\mathrm{SU}(n)$-equivariant. We note that, as $g \in \mathrm{SU}(n)$ we have $g^{-t} = \bar{g}$, so that, $\mathrm{SU}(n)$ acts on $\mathbb{C}^{*n}$ by $\bar{g}$. Finally we see

$$\prod_{i=1}^n (\sum_j g_{ij}z_j)^{s_i} = \sum g_s \prod_i z_i^{s_i'} \mapsto \sum g_s \otimes_i e_i^{\otimes s_i'} = \otimes_i (ge_i)^{\otimes s_i}.$$

Similarly, we work out the dual action. This proves the natural isomorphism.

We use the isomorphism to calculate the characters. Hence for $t \in T \cap \mathrm{SU}(n)$

$$\chi(t) = \left(\sum_{i_1 \le \dots i_p} t_{i_1} \dots t_{i_p}\right)\left(\sum_{j_1 \le \dots j_q} t_{j_1}^{-1} \dots t_{j_q}^{-1}\right).$$

Thus the weights form the set

$$\{\lambda \in (\mathbb{Z} \cap [-q,p])^n \mid \sum_i \lambda_i = p - q\}/\mathbb{Z}(1,\dots,1).$$

(3) It is easy to see that Δ has image in $V_{p-1,q-1}$ just by differentiation and surjectivity follows form "integrating", and from the fact that differentiation and integration preserve homogeneity. Equivariance can be checked by explicit calculation, as previous.

(4) If $f \in H_{p,q}$ then from the previous part

$$\Delta(\pi(g)f) = \pi(g)\Delta(f) = 0.$$

Hence, $H_{p,q}$ is invariant.

As $H_{p,q}$ is the kernel of the surjection $V_{p,q} \to V_{p-1,q-1}$. So its dimension is

$$\begin{aligned}\dim(V_{p,q}) - \dim(V_{p-1,q-1}) &= \binom{n+p-1}{p}\binom{n+q-1}{q} - \binom{n+p-2}{p-1}\binom{n+q-2}{q-1} \\ &= (n-1)(p+q+n-1)\frac{(n+p-2)!\,(n+q-2)!}{p!\,q!\,(n-1)!^2}.\end{aligned}$$

Highest weight is $(p, 0, \ldots, -q)$. Using Weyl character formula we check that the dimension of the representation attached to the highest weight is

$$\frac{p+q+n-1}{n-1} \prod_{i=2}^{n-1} \frac{n-i+q}{n-i} \prod_{j=2}^{n-1} \frac{p-1+j}{j-1}$$
$$= (n-1)(p+q+n-1) \frac{(n+p-2)!\,(n+q-2)!}{p!\,q!\,(n-1)!^2}.$$

Thus $H_{p,q}$ is irreducible.

(5) Let π irreducible representation of $\mathrm{SU}(3)$ with highest weight of the form $(p, q, r)/\mathbb{Z}(1, 1, 1)$ which is same as $(p-q, 0, r-q)$. Thus π is isomorphic to $H_{p-q,r-q}$. Similar happens for $\mathrm{SU}(2)$. Finally if $n \geq 4$ then there exists irreducible representation with highest weight $\lambda/Z(1, 1, 1, 1)$ with $\lambda_2 \neq \lambda_3$, thus clearly can not be isomorphic to any $H_{p,q}$.

Exercise 9. *For a compact group G, prove that any subrepresentation (i.e., closed irreducible subspace) of the representation of $G \times G$ on $L^2(G)$ is of the form $E_\Pi := \hat{\oplus}_{\pi\in\Pi}\mathcal{A}(\pi)$ for some subset Π of $\mathrm{Irr}(G)$. Give an analogous description for subrepresentations of (say) the right regular representation of G on $L^2(G)$.*

Solution 9. Let $(\rho, V) \subseteq (\rho_0, L^2(G))$ be a subrepresentation of $G \times G$. Using an isotypic decomposition of V (theorem 4.11) we get

$$V = \hat{\oplus}_{\sigma\in\mathrm{Irr}(G\times G)} V[\sigma],$$

where $V[\sigma]$ is the σ isotypic component of V. Thus it is enough to show that if $V[\sigma] \neq 0$ then $\exists \pi \in \mathrm{Irr}(G)$ such that $\mathcal{A}(\pi) = V[\sigma]$. We recall (Lemma 3.3) that $\mathcal{A}(\pi) \cong \pi \boxplus \tilde{\pi}$, and the Peter-Weyl theorem

$$L^2(G) = \hat{\oplus}_{\pi\in\mathrm{Irr}(G)}\mathcal{A}(\pi).$$

We also recall that $\pi(\alpha_\tau)$ is a projector on the τ-isotypic space for $\alpha_\tau = \dim(\tau)\overline{\chi_\tau}$ and $\pi \in \mathrm{Irr}(G)$ (theorem 4.9). Now let $\sigma = \pi_1 \boxtimes \pi_2$. But $\rho_0(\alpha_\sigma) \mid_{\mathcal{A}(\pi)} = 0$ unless $\pi_1 = \pi$ and $\pi_2 = \tilde{\pi}$, in particular, $\pi_2 = \tilde{\pi}_1$. In other words, $L^2(G)[\sigma] = \mathcal{A}(\pi)$ if $\sigma = \pi \boxtimes \tilde{\pi}$ and zero otherwise. But $\rho(\alpha_\sigma) = \rho_0(\alpha_\sigma) \mid_V$, which concludes the proof of the claim.

Exercise 10. *Let G be a unimodular locally compact group and π a compact-type unitary representation of G. Show that*

$$\pi \cong \hat{\oplus}\pi_j,$$

where π_j are irreducible and $\#\{k \mid \pi_j \cong \pi_k\} < \infty$, for all j. [A proof is recorded in the lecture notes at the end of §4, so the homework problem is basically to study and rewrite that proof.]

Solution 10. First we show that there exists at least one irreducible subspace of π. Let $f \in C_c(G)$ with $f(g) = \overline{f(g^{-1})}$, so that $\pi(f)$ is a self-adjoint, nonzero, compact operator. Thus by the psectral theory $\pi(f)$ has an eigenvalue, call λ, with eigenvector v. We consider the cyclic representation $\overline{\langle v\rangle}$ generated by v. Let $\overline{\langle v\rangle}$ is not irreducible and has a orthogonal decomposition of the form $\overline{\langle v\rangle} = V \oplus V^\perp$ with $v = v_1 + v_1'$, where V and its orthogonal compliment are G-invariant. So they are also $\pi(f) - \lambda$ invariant. This implies that v_1 and v_1' are also $\pi(f)$ eigenvectors with eigenvalue λ. We confirm that none of the v_i are zero. Because otherwise V would lie in a proper subspace of $\overline{\langle v\rangle}$ contrary to the construction. Thus if V_λ is the λ eigenspace of $\pi(f)$ then we obtained

$$V_\lambda \cap \overline{\langle v_1\rangle} \subsetneq V_\lambda \cap \overline{\langle v\rangle}.$$

As LHS has a strictly smaller dimension than the RHS and the RHS is a finite dimensional space after finitely many steps we will obtain a nonzero irreducible subspace.

By Zorn's lemma we can find a maximal collection of mutually orthogonal collection $\{\pi_j\}$ of irreducible subrepresentation. We claim that $\oplus\pi_j$ is dense in π. If not we can take the orthogonal compliment of the above which would satisfy the same hypothesis without any irreducible subrepresentation, contradicting the previous argument.

For π_j we choose a self adjoint compact integral operator $\pi(f)$ having an eigenvector in π_j. if $\pi_j \cong \pi_k$ then $\pi(f)$ will have an eigenvector in π_k with the same eigenvalue. But the eigenspaces of $\pi(f)$ are finite dimensional, which concludes the proof of the claim.

Exercise 11. *Let G be a unimodular Lie group, equipped with some Haar measure dg. Let (π, V) be a Hilbert representation. Say that $v \in V$ is a* smooth vector *if the map $G \to V$ given $g \mapsto gv$ is smooth (i.e., infinitely differentiable, with the same definition as for scalar-valued functions). Show that:*

(1) *For $f \in C_c^\infty(G)$, the map $G \to L^1(G)$ given by $g \mapsto [x \mapsto f(g^{-1}x)]$ is smooth.*
(2) *For each $v \in V$ and $f \in C_c^\infty(G)$, the vector $\pi(f)v$ is smooth.*
(3) *The space of smooth vectors is dense in V.*

Solution 11. (1) Let $X \in \mathfrak{g} := \mathrm{Lie}(G)$. Then the induced action by X on f is given by

$$Xf : x \mapsto \partial_{t=0} f(\exp(-tX)x).$$

As f is smooth the Xf exists and f being compact supported confirms that $Xf \in L^1(G)$.

(2) Note that the composed map $G \to L^1(G) \to V$ given by $g \mapsto f(g^{-1}.) \mapsto \pi(f(g^{-1}.))$ is smooth, as the first map is smooth by (1) and the second map is Lipschitz. Now by differentiating under the integral sign and a change of variable

$$\begin{aligned} X\pi(f)v &= \partial_{t=0} \int_G f(x)\pi(\exp(tX)x)v dg \\ &= \int_G \partial_{t=0} f(\exp(-tX)x)\pi(x)v dg = \pi(Xf)v. \end{aligned}$$

The last equality holds from (1) and implies that v is differentiable, hence smooth by repeating the same argument.

(3) Let $\epsilon > 0$. Note that for any $v \in V$ there exists a neighbourhood U of the identity such that by the continuity of the representation we can obtain

$$\|\pi(u)v - v\|_V < \epsilon, \quad \forall u \in U.$$

By C^∞ Urysohn's lemma there exists $f \in C_c^\infty(U)$ with $\|f\|_1 = 1$. Then

$$\|\pi(f)V - v\| \leq \int_U |f(g)| \|\pi(g)v - v\| < \epsilon.$$

But $\pi(f)v$ is smooth by (2), hence we conclude.

Exercise 12. *Let G be a reductive complex algebraic group, and V a finite-dimensional vector space. Show that any holomorphic representation $G \to \mathrm{GL}(V)$ is algebraic. [You may use that if Ω is a connected open subset of $\mathbb{C}^n$ that intersects $\mathbb{R}^n$, then any holomorphic function $\Omega \to \mathbb{C}$ that vanishes on $\mathbb{R}^n$ is identically zero.]*

Solution 12. We start by recalling the Caratn decomposition $G = K\exp(\mathfrak{p})$. Note that if π is a holomorphic representation of G then for $g = k\exp(X)$ we have

$$\pi(g) = \pi(k)\pi(\exp(X)) = \pi(k)\exp(d\pi(X)) = \pi(k)\exp(id\pi(X/i)),$$

where $d\pi$ is the differentiated representation of $\mathrm{Lie}(G)$ which is $\mathbb{C}$-linear if π is holomorphic, and the second equality follows from the uniqueness of one parameter flow with given velocity. Now we recall that $\mathfrak{p} = i\mathfrak{k}$, thus $X/i \in \mathfrak{k}$. In other words, $\pi \mapsto \pi \mid_K$ is injective. Finally, we recall that $\mathbb{C}[G] \cong \mathcal{A}(K)$, in particular, algebraic representations of G are in bijection with finite dimensional representation of K (theorem 5.13). As V is finite dimensional there exists an algebraic representation τ of G such that $\tau \mid_K = \pi \mid_K$. But the injectivity implies that $\pi \cong \tau$, hence algebraic.

Exercise 13. *Let $G_{\mathbb{C}}$ be a complex reductive algebraic group, embedded in $\mathrm{GL}_n(\mathbb{C})$ in such a way that $\Theta(G_{\mathbb{C}}) = G_{\mathbb{C}}$. Set $G := G_{\mathbb{C}} \cap \mathrm{GL}_n(\mathbb{R})$ and $K := G_{\mathbb{C}} \cap \mathrm{O}(n)$. The groups G that arise in this way turn out to be the* reductive real algebraic groups*. Extend the Cartan decomposition, as proved in lecture for reductive complex algebraic groups, to the real case, as follows.*

1. *Set $\mathfrak{k} := \mathrm{Lie}(K)$, $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{p} := \{x \in \mathfrak{g} : x^t = x\}$. Show that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and that the map $K \times \mathfrak{p} \to G$ is a diffeomorphism.*
2. *Take $p, q \geq 1$ and $G_{\mathbb{C}} := \mathrm{U}(p,q)$. Show that $G = \mathrm{O}(p,q)$ and $K = \mathrm{O}(p) \times \mathrm{O}(q)$. Describe $\mathfrak{k}$, $\mathfrak{p}$ and the decomposition of $\mathfrak{g}$ explicitly. Assuming the fact that $\mathrm{O}(n)$ has two connected components, show that G has four connected components. Can you describe these explicitly?*

Solution 13.

(1) Let $\theta := d\Theta$, so that, $\theta(\mathrm{Lie}(G_{\mathbb{C}})) = \mathrm{Lie}(G_{\mathbb{C}})$. Also $\mathfrak{g} = \mathrm{Lie}(G_{\mathbb{C}}) \cap \mathfrak{gl}_n(\mathbb{R})$ and $\theta \mid_{\mathfrak{g}} : x \mapsto -x^t$. We also note that, $\mathfrak{k} := \{x \in \mathfrak{g} \mid x^t = -x\}$ as $\mathfrak{k} = \mathrm{Lie}(G_{\mathbb{C}}) \cap \mathfrak{o}(n)$. As θ is an involution we can decompose $\mathfrak{g}$ in ± 1 eigenspace of θ. In particular, we can write

$$\mathfrak{g} \ni x = x^+ + x^-, \quad x^\pm := \frac{x \pm \theta(x)}{2}.$$

Hence, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

G, K, and $\mathfrak{p}$ are intersections of smooth manifolds with closed subsets. So the restriction of the corresponding diffeomorphism in $G_{\mathbb{C}}$ would yield a diffeomorphism in the real group.

(2) Recall that, for $J := \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$

$$U(p,q) := \{g \in \mathrm{GL}_{p+q}(\mathbb{C}) \mid g^* J g = J\}.$$

Clearly,

$$U(p,q) \cap \mathrm{GL}_{p+q}(\mathbb{R}) = \{g \in \mathrm{GL}_{p+q}(\mathbb{R}) \mid g^t J g = J\} = O(p,q).$$

We write an element $g \in K$ in $p+q$ block matrix form, that is, $k = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathrm{Mat}_{p\times p}$, $B \in \mathrm{Mat}_{p\times q}$, etc.. As $k \in O(p+q) \cap O(p,q)$ we have

$$k^t k = I_{p+q}, \quad k^t J k = J,$$

we obtain

$$A^t A = I_p, D^t D = I_q, B^t A = 0 = D^t C \implies A \in O(p), D \in O(q), B = 0 = C.$$

Thus $k \in O(p) \times O(q)$. The reverse inclusion is obvious.

By differentiating we obtain

$$\mathfrak{g} = \{X \in \mathfrak{gl}_{p+q}(\mathbb{R}) \mid X^t J + JX = 0\} =: \mathfrak{o}(p,q).$$

From the description as θ eigenspaces we can calculate

$$\mathfrak{k} = \{X \in \mathfrak{gl}_{p+q}(\mathbb{R}) \mid JXJ = -X^t = X\}.$$

Doing a similar calculation as the above one obtains

$$\mathfrak{k} = \mathfrak{o}(p) \times \mathfrak{o}(q).$$

Similarly,

$$\mathfrak{p} = \{\begin{pmatrix} & X \\ -X^t & \end{pmatrix} \mid X \in \mathrm{Mat}_{p\times q}(\mathbb{R})\}.$$

Using Cartan decomposition, $K = O(p) \times O(q)$ is the deformation retract of $G = O(p,q)$. Thus K has same number of connected component as G, which is 4. They are given by $K_i \exp(\mathfrak{p})$, where K_i are of form $O(p)^\circ \times O(q)^\circ$, $O(p)^1 \times O(q)^\circ$, $O(p)^\circ \times O(q)^1$, and $O(p)^1 \times O(q)^1$, where $O(n)^1 = O(n) \setminus O(n)^\circ$.

Exercise 14. *Following the hint given in lecture, show that for any compact connected Lie group K and torus $S \leq K$, the centralizer $Z_K(S)$ is the union of all the maximal tori containing S.*

Solution 14. We want to show that

$$Z_K(S) = \bigcup_{\text{torus } T \supseteq S} T.$$

Note that, "$\supseteq$" is obvious; as T is abelian and contains S, T clearly centralizes S. We prove "$\subseteq$". Let $g \in Z_K(S)$. This implies that $S \subset Z_K(g)^\circ$ as S is connected. Let $S' \subseteq Z_K(g)^\circ$ be a maximal torus containing S. by theorem 6.21 we have $Z(Z_K(g)^\circ) \subset S'$. But from theorem 6.19 we know that $g \in Z_K(g)^\circ$, hence $g \in Z(Z_K(g)^\circ)$. Choosing T to be a maximal torus of K which contains S', we conclude.

Exercise 15. *Let K be the compact connected group $\mathrm{SO}_N(\mathbb{R})$. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^N$.*

(1) *Give a description of the complexified Lie algebra $\mathfrak{g}$ of K.*

(2) *Show that with a maximal torus $T \leq K$ can be represented by block diagonal matrices $(t_1, \ldots, t_n, [1])$ where each t_k is a 2×2 matrix of form*

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R},$$

where $[1]$ denotes 1×1 block which arises only when $N = 2n+1$ odd.

(3) *Compute the root space decomposition of $\mathfrak{g}$. For a given root α compute $X_\alpha, Y_\alpha, H_\alpha$, and the root reflection s_α.*

(4) *Compute the Weyl group W generated by s_α. Check that that indeed coincides with $N(T)/T$.*

Solution 15. A summary of the solution is given here. We realize $\mathrm{SO}_N(\mathbb{R})$ acting on $\mathbb{R}^N$ equipped with the standard inner product. We check that T is indeed a torus (compact, connected, abelian), in fact, is homeomorphic to $(S^1)^r$ for $r = [N/2]$ when θ is restricted to $\mathbb{R}/2\pi\mathbb{Z}$. We want to show that it is a maximal torus. It is enough to show that $Z_K(T) \subseteq T$. Let $A \in Z_K(T)$. We consider an element $t_k := \mathrm{diag}(1, 1, \ldots, -1, -1, \ldots, 1, 1)$ where $(-1, -1)$ lies in $(2k-1, 2k)$th position. Let $Ae_{2k-1} = \sum_{i=1}^N a_i e_i$. Then

$$t_k A e_{2k-1} = a_1 e_1 + \cdots - a_{2k-1} e_{2k-1} - a_{2k} e_{2k} + \cdots + a_N e_N,$$

and

$$A t_k e_{2k-1} = -A e_{2k-1} = -a_1 e_1 - \cdots - a_N e_N.$$

Hence $a_i = 0$ for $i \neq 2k-1, 2k$. In other words A is a rotation in the plane generated by $e_{2k-1} e_{2k}$, thus must be of the form t_k. So T is a maximal torus.

Now we complexify T and K, and denote them H and G respectively. We will now describe H. We construct a new basis

$$f_j := \frac{e_{2j-1} + e_{2j}}{\sqrt{2}}, f_{n+j} := \frac{e_{2j-1} - e_{2j}}{\sqrt{2}}; \quad 1 \leq j \leq n; \qquad (f_{2n+1} = e_{2n+1}).$$

The above basis is an eigenbasis and digaonalize T over $\mathbb{C}$ to

$$\{\mathrm{diag}(e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_n}, [1]) \mid \theta_i \in \mathbb{C}\}.$$

For future purpose we define H to be a permutation of the above

$$H := \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_1}, \ldots, e^{-i\theta_n}, [1]\},$$

which can be obtained by conjugating by a permutation matrix from $\mathrm{SO}_N(\mathbb{C})$.

We now note that the complexified Lie algebra$\mathfrak{g} := \mathfrak{so}_N(\mathbb{C})$ can be written in the split form as

$$\mathfrak{so}_N(\mathbb{C}) := \{X \in M_N(\mathbb{C}) \mid X^t J + JX = 0\},$$

where

$$J = \begin{cases} \begin{pmatrix} & I_n \\ I_n & \end{pmatrix}, & \text{if } N = 2n \\ \begin{pmatrix} & I_n & \\ I_n & & \\ & & 1 \end{pmatrix}, & \text{if } N = 2n+1 \end{cases}.$$

For instance, a generic matrix in $\mathfrak{so}_{2n+1}(\mathbb{C})$ is given by $\begin{pmatrix} a & b & x \\ c & -a^t & y \\ -y^t & -x^t & 0 \end{pmatrix}$ with $a, b, c \in M_n(\mathbb{C})$ and $x, y \in \mathbb{C}^n$, and $b^t = -b$, $c^t = -c$.

Now we compute the root space decomposition. We have to compute the action of $\mathrm{ad}(H)$ for $H \in \mathfrak{h}$ where

$$\mathfrak{h} := \{H := \mathrm{diag}(\theta_1, \dots, \theta_n, -\theta_1 \dots, -\theta_n, [0]) \mid \theta_i \in \mathbb{C}\}.$$

We fix a basis of h to be $\{E_{i,i} - E_{n+i,n+i}\}$. Let the dual basis of $\mathfrak{h}^*$ is given by $\{\lambda_i\}$ such that that $\theta_i = \lambda_i(H)$. Now we calculate the eigenvectors of $\mathrm{ad}(E_{i,i} - E_{n+i,n+i})$. We list the roots $\alpha \in \Phi$, $X_\alpha \in \mathfrak{g}^\alpha$, $Y_\alpha \in \mathfrak{g}^{-\alpha}$, and $H_\alpha = [X_\alpha, Y_\alpha]$.

$$\Phi = \{\pm(\lambda_j \pm \lambda_k) \mid j < k\} \sqcup \{\pm\lambda_j\},$$

where the second part arises when $N = 2n+1$.

α	X_α	Y_α	H_α
$\lambda_j - \lambda_k$	$E_{j,k} - E_{n+k,n+j}$	$E_{k,j} - E_{n+j,n+k}$	$E_{j,j} - E_{k,k} - E_{n+j,n+j} + E_{n+k,n+k}$
$\lambda_j + \lambda_k$	$E_{j,n+k} - E_{k,n+j}$	$E_{n+k,j} - E_{n+j,k}$	$E_{j,j} + E_{k,k} - E_{n+j,n+j} - E_{n+k,n+k}$
$-\lambda_j - \lambda_k$	$E_{n+j,k} - E_{n+k,j}$	$E_{k,n+j} - E_{j,n+k}$	$-E_{j,j} - E_{k,k} + E_{n+j,n+j} + E_{n+k,n+k}$
λ_j	$E_{j,2n+1} - E_{2n+1,n+j}$	$2(E_{2n+1,j} - E_{n+j,2n+1})$	$2E_{j,j} - 2E_{n+j,n+j}$
$-\lambda_j$	$E_{2n+1,j} - E_{n+j,2n+1}$	$2(E_{j,2n+1} - E_{2n+1,n+j})$	$-2E_{j,j} + 2E_{n+j,n+j}$

The last two rows appear only when $N = 2n+1$. A positive system of roots can be given by

$$\Phi^+ = \{\lambda_j \pm \lambda_k \mid j < k\} \sqcup \{\lambda_j\},$$

where the second part appears only when $N = 2n+1$. simple system of roots is given by

$$\Delta = \{\lambda_j - \lambda_{j+1} \mid 1 \le j \le n, \lambda_{n+1} = 0, -\lambda_{n-1}, \text{ for } N \text{ odd or even, resp.}\}.$$

The root space decomposition can be checked by means of the formula

$$\mathrm{ad}(H)X_\alpha = \alpha(H)X_\alpha, \quad \mathrm{ad}(H)Y_\alpha = -\alpha(H)Y_\alpha.$$

Let us choose following "standard" chamber $C \subseteq \mathfrak{h}_{\mathbb{R}}$:

$$C := \{\mathrm{diag}(\theta_1, \dots, \theta_n, -\theta_1 \dots, -\theta_n, [0]) \mid \theta_1 > \dots > \theta_n [> 0]\}.$$

With respect to this chamber the positive roots in Φ would be

$$\lambda_j \pm \lambda_k, \quad 1 \le j < k \le n,$$

for $N = 2n$, and

$$\lambda_j \pm \lambda_k, \lambda_k, \quad 1 \le j < k \le n,$$

for $N = 2n+1$. We claim that Δ as described above is a system of simple roots. First we check that they are indeed simple root. Then we check that Δ is linearly independent: If $\sum c_i(\lambda_i - \lambda_{i+1}) = 0$, then $c_i = 0$ invoking linear Independence of λ_i. Finally, we check that Φ^+ can be obtained by a $\mathbb{Z}_{\geq}0$ span of Δ. To check this we note (for $N = 2n+1$), for example,

$$\lambda_j + \lambda_k = (\lambda_j - \lambda_j + 1) + \cdots + (\lambda_{k-1} - \lambda_k) + 2(\lambda_k - \lambda_{k+1}) + \cdots + 2\lambda_n.$$

Rest of them can be proved similarly.

Now we turn to find the reflections s_α for $\alpha \in \Phi$. Recall the map F_α whose differential dF_α sends $\begin{pmatrix} & 1 \\ & \end{pmatrix}$ to X_α so that $s_\alpha = F_\alpha(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix})T$. Note that

$$\begin{aligned} F_\alpha(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}) &= F_\alpha(\begin{pmatrix} \cos \pi/2 & \sin \pi/2 \\ -\sin \pi/2 & \cos \pi/2 \end{pmatrix}) = F_\alpha(\exp \begin{pmatrix} & \pi/2 \\ -\pi/2 & \end{pmatrix}) \\ &= \exp(\frac{\pi}{2} dF_\alpha(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix})) = \exp(\frac{\pi}{2}(X_\alpha - Y_\alpha)). \end{aligned}$$

The Weyl group is generated by $\{s_\alpha\}_{\alpha \in \Delta}$.

Now we construct the Weyl group analytically. We claim the following:

- For $N = 2n+1$, $W \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$.
- For $N = 2n$, $W \cong H_n \rtimes S_n$, where H_n is the hyperplane in $(\mathbb{Z}/2\mathbb{Z})^n$ defined by $\sum \epsilon_i = 0$.

First we construct the map explicitly. We understand $\mathbb{C}^N$ as complexification of the standard representaion of $\mathrm{SO}_N(\mathbb{R})$ on $\mathbb{R}^N$. We checked that the torus T acts on $\mathbb{C}^N$ by distinct characters $e^{\pm i\theta_j}$ for $j = 1..n$. Thus any $g \in N_K(T)$ permutes the eigenlines. But the action of g is defined on $\mathbb{R}$ so it preserves the complex conjugation, in the sense that, if g maps $e^{i\theta}$ to $e^{i\theta'}$ then it should map $e^{-i\theta}$ to $e^{-i\theta'}$. Thus $N_K(T)$ naturally acts as a permutation on $\{\pm 1, \ldots, \pm n\}$ such that $\sigma(-k) = -\sigma(k)$. We also have permutations in n pairs of $\{-j, j\}$. Group of such permutation is $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$.

Now we show the injectivity. Let g acts trivially on every eigenline. As g preserves every one dimensional subspace it must be diagonal. Thus it commutes with T. Hence $g \in Z_K(T)$. But maximality of T implies that $Z_K(T) = T \ni g$. Thus the map is injective.

To show surjectivity we need divide into parities. Let $N = 2n+1$ first. For given (σ, ϵ), where $\sigma \in S_n$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n$, we first choose the permutation matrix which permute the eigenplanes $P_j := \mathbb{R}(e_{2j-1} \oplus + e_2 j)$ to p_k such that $\sigma(j) = k$. Then we transpose between a line and its conjugate line according to epsilon. We need to to make sure that this matrix lie in SO_{2m+1}, that is, it has $\det = 1$. For that we choose action on $\mathbb{R}e_{2n+1}$ by $(-1)^{\sum} \epsilon_i$. Clearly this matrix lie in $N_K(T)$. We prove the even case similarly. Because the codomain only contains ϵ with $\sum \epsilon_i = 0$ the $\det = 1$ property is immediate.

Exercise 16. *Say that a pair of chambers C, C' are* adjacent *if they are separated by exactly one root hyperplane $\beta^\perp$; in other words, there is a root β so that*

- $\beta(C) > 0$,
- $\beta(C') < 0$, *and*
- $\alpha(C)$ *and* $\alpha(C')$ *have the same sign for any* $\alpha \in \Phi - \mathbb{Q}\beta$.

We say in that case that β is a wall *of C and also of C'.*

(1) *Interpret Lemma* 6.39 *in terms of these notions.*
(2) *Show that if the chambers C, C' are adjacent with common wall $\beta^\perp$, then $s_\beta C = C'$ and $s_\beta C' = C$.*
(3) *Let $\beta \in \Phi^+(C)$. Show that the following are equivalent:*
   (a) *$\beta^\perp$ is a wall of C (in the sense defined above).*
   (b) *$\beta \in \Delta(C)$.*
   (c) *$\beta^\perp \cap \overline{C}$ contains a nonempty open subset of the hyperplane $\beta^\perp$.*
(4) *With notation and assumptions as in the conclusion of Lemma 6.39, show that there exist $\beta_1, \dots, \beta_n \in \Delta(C)$ so that $C = s_{\beta_j} \cdots s_{\beta_1} C_j$ for $j = 0..n$.*
(5) *Show that for any Weyl chamber C, the root reflections s_β taken over $\beta \in \Delta(C)$ generate W.*

Solution 16. (1) Left for the reader.

(2) Let $x \in C$. Then
$$\beta(s_\beta(x)) = \beta(x) - \beta(x)\beta(H_\beta) = \beta(x) < 0.$$
We know that s_β reflects with respect to $\beta^\perp$ and s_β is continuous so $s_\beta(C) \subseteq C'$. Similarly, $s_\beta(C') \subseteq C$. Thus $s_\beta(C) = C'$ and $s_\beta(C') = C$.

(3) $(a) \implies (b)$: By the part (1) if $\beta^\perp$ is a wall of C then C and $s_\beta(C)$ are adjacent. So that $\beta^\perp$ is the only root hyperplane between C and $s_\beta(C)$ such that the three conditions of the adjacency hold. Let $\beta(C) > 0$ and $\beta = \alpha_1 + \alpha_2$ where $\alpha_i \neq \mathbb{Q}\beta$. Then $\alpha_i(C) > 0$ and $\alpha_i(s_\beta(C)) > 0$ which contradicts that $\beta(s_\beta(C)) < 0$.
$(b) \implies (c)$: Let $\beta \in \Delta(C)$. We claim that there exists $x \in \overline{C}$ so that $\beta(x) = 0$ but $\gamma(x) > 0$ for all $\gamma \in \Delta(C) - \{\beta\}$. Then x admits a small convex neighborhood U that doesn't intersect any root hyperplane other than $\beta^\perp$. Moreover, $s_\beta U \cap U$ contains x, hence is nonempty. We may assume also that U is small enough that $\gamma(U) > 0$ for all $\gamma \in \Delta(C) - \{\beta\}$, hence that $U \cap \beta^\perp \subseteq \overline{C}$. But $U \cap \beta^\perp$ is open in U, so this gives (c).
$(c) \implies (a)$: We essentially follow the proof of Lemma 6.39.

(4) We induct on j. Note that $C'' := s_{\beta_j} \cdots s_{\beta_1} C_{j+1}$ is adjacent to C_0. Let $\beta_{j+1}^\perp$, with $\beta_{j+1} \in \Delta(C_0)$, be their common wall. Then $s_{\beta_{j+1}} C'' = C_0$, as required.

(5) We know that W acts freely on the Weyl chambers. As from part (4) we conclude that for given $C, C' = w(C), w \in W$ Weyl chambers, there exist $\alpha_i \in \Delta(C)$ such that $s_{\alpha_1} \dots s_{\alpha_j}(C) = C'$. Hence $w = s_{\alpha_1} \dots s_{\alpha_j}$.

Exercise 17. *A* root system *is a finite dimensional Euclidean space* $(V, \langle , \rangle)$*, equipped with a spanning subset* $\Phi \subset V$ *s.t.* $\forall \alpha \in \Phi$*,* $s_\alpha : V \to V$ *defined by* $v \mapsto v - 2\frac{\langle \alpha, v\rangle}{\langle \alpha,\alpha\rangle}$*, satisfies* $s_\alpha(\Phi) = \Phi$ *and* $2\frac{\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$ *for* $\alpha, \beta \in \Phi$*. A root system is called* reduced *if* $\mathbb{C}\alpha \cap \Phi = \{\pm\alpha\}$.

(1) *Show that if* $(V, \langle , \rangle, \Phi)$ *is a (reduced) root system then* $(X, \Phi, \check{X}, \check{\Phi})$ *is a (reduced) root datum, where either*

Simply Connected: *Weight lattice:*

$$X := \{\lambda \in V \mid \langle \lambda, \alpha\rangle \in \mathbb{Z}, \forall \alpha \in \phi\},$$

and coroot lattice:

$$\check{X} := \sum_{\alpha\in\Phi} \mathbb{Z}\check{\alpha}, \quad \check{\alpha} = 2\frac{\langle \alpha, \dot{\rangle}}{\langle \alpha, \alpha\rangle}.$$

Adjoint: *Root lattice:*

$$X := \sum_{\alpha\in\Phi} \mathbb{Z}\alpha,$$

and coweight lattice:

$$\check{X} := \{z \in V^* \mid z(\alpha) \in \mathbb{Z}, \forall \alpha \in \Phi\}.$$

(2) *Show that there exists a unique (up to isomorphism) simply connected compact group with root system of* G_2 *type.*