Riemann surface and Hurwitz theory

Textbook, office hour
April 5th off,
alternate day/time
HW are all exercises in textbook

Outline of the course:

- Preliminaries
  complex analysis   general topology
  Manifold theories
  (Algebraic topology)
- Examples of Riemann surfaces
  (Classification of compact R.S.)
- Theory of R.S. < local structure of maps / Riemann-Hurwitz Formula
- Some algebraic topology
  (Fundamental gp, covering space)
- Hurwitz theory — "map counting"

(Hyenho's toric var.) UZH's course MAT 258

Goal

↳ Monodromy representation
Degeneration formula
Hurwitz potential
...

Overview

~~Hurwitz~~

Riemann surfaces : 2-dimensional manifolds
(with complex structure)
↳ locally isomorphic to $\mathbb{C}^n$

There are maps between R.S.

Hurwitz theory :

Given some topological constraints (genus, degree, etc.),
$\boxed{\text{count}}$ the number of $\boxed{\text{maps}}$ between
Riemann surface.

maps : (ramified) "coverings"

think of the action of covering

In order to cover :

- Your cover has to fit the surface
  (local isomorphism)

- Your can have a few layers on the top
  (# of layers : degree)

~~Count~~

(can't explain ramifications for now,
I would perhaps say they are ~~wrinkles~~, but parts on the sheet that get sticky
it's ambiguous.)

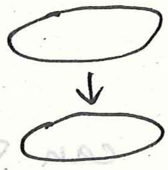
count : • weighted count

• ~~up to isomo~~ insensitive to ~~precise shape~~ streching
(up to isomorphism)

Question 1:

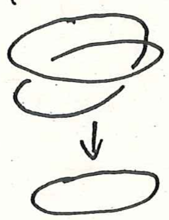
How many connected coverings can S^1 have?

- 1 layer (= degree 1)



weight 1

- 2 layers (= deg 2)



weight $\frac{1}{2}$

- 3 layers (= deg 3)

weight $\frac{1}{3}$

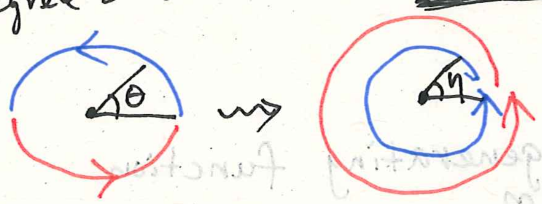
weight: keep track of the ambiguity of the map (automorphism)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sigma"'] & S^1 \arrow[d, equal] \\ X \arrow[r, "f"] & S^1 \end{tikzcd}$$

If σ is invertible, and the diagram commutes, $\sigma \in \mathrm{Aut}(X \xrightarrow{f} S^1)$

$\boxed{\text{Define weight to be } \dfrac{1}{|\mathrm{Aut}(X \xrightarrow{f} S^1)|}}$

E.g. degree 2 cover:



$\eta = 2\theta$

possible automorphisms:
- Id
- $\theta \mapsto \theta + \frac{\pi}{2}$

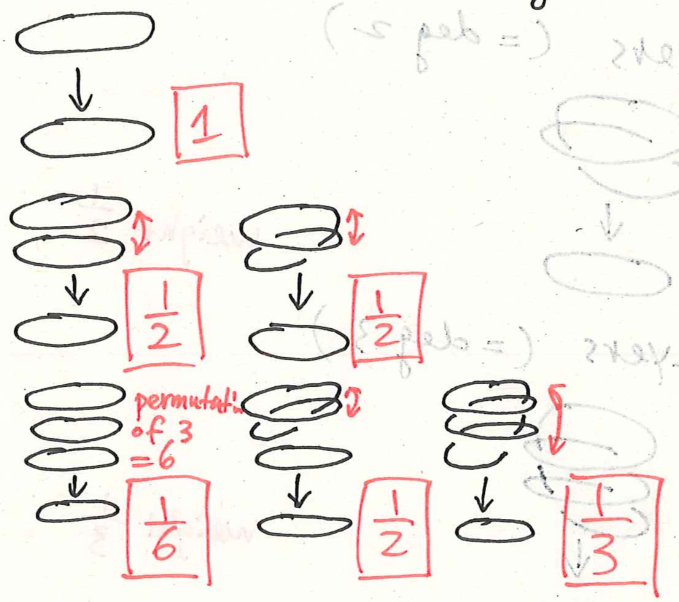
The weighted count: connected Hurwitz number

$$H_d = \frac{1}{d}$$

Question 2

How many possibly disconnected covering can S' have?

- deg 1
- deg 2
- deg 3



The weighted count: disconnected Hurwitz number

$$H^{\bullet}_d$$

Thm: $H^{\bullet}_d = 1 \quad \forall\, d \geq 1$

Idea of proof: Form a generating function

$$f(x) = \sum_{d \geq 1}^{\infty} H_d x^d$$

Claim: $\sum_{d\geq 1}^{\infty} H^{\bullet}_d x^d = \exp\left(\sum_{d\geq 1}^{\infty} H_d x^d\right)$

Why? Think of the Taylor expansion

$$\exp\left(\sum_{i\geq 1} H_d x^d\right) = 1 + \left(\sum_{i\geq 1} H_d x^d\right) + \frac{\left(\sum_{i\geq 1} H_d x^d\right)^2}{2!} + \cdots$$

Expand and collect x^i coefficients.

Leave it as an exercise.

Now, $\sum_{d\geq 1} H_d x^d = \sum_{i\geq 1} \frac{1}{d} x^d = \log\left(\frac{1}{1-x}\right)$

$$\sum_{d\geq 1} H^{\bullet}_d x^d = \exp\left(\log\left(\frac{1}{1-x}\right)\right) = \frac{1}{1-x} = 1 + x + x^2 + \cdots$$

BOOM!

Maybe try $d=3$ in the claim

x^3 coefficient:

$$H_3 + \frac{1}{2!}\cdot 2H_1H_2 + \frac{1}{3!}\cdot H_1^3$$

from 1st term | 2nd term | 3rd term

deg 3 connected covering | deg 2 conn. cov., deg 1 .. | deg 1, deg 1, deg 1

Show me a good explanation next week to get reward!

Why coefficient 1? b/c you can't swap the two components as a whole.

why coeff. - b/c you CAN swap the th components a a whole! A total of ways!

Start of serious stuff

Complex analysis

Def: A function $f: \mathbb{C} \to \mathbb{C}$ is ~~differentiable~~ or holomorphic ← I prefer this term if and only if the limit

$$\lim_{|h|\to 0} \frac{f(z_0+h)-f(z_0)}{h} \quad \text{exists}$$

Write $f'(z_0) = \lim_{|h|\to 0} \frac{f(z_0+h)-f(z_0)}{h}$

A function f is differential at a domain $U \subseteq \mathbb{C}$ if it is differential at $\forall\, u \in U$

E.g: $f(z) = z^2$

$$\boxed{\begin{aligned} & f(x+iy) = (x+iy)^2 = x^2 + 2ixy - y^2 \\ & h = \Delta x + i\Delta y \\ & \frac{f(x+iy+\Delta x+i\Delta y) - f(x,y)}{\Delta x + i\Delta y} = \cdots \text{Brutal force!} \end{aligned}}$$

or. $\frac{f(z+h)-f(z)}{h} = \frac{(z+h)^2 - z^2}{h} = \frac{h(2z+h)}{h} = 2z + h$

$|h| \to 0$

$f'(z) = 2z$

$\lim_{|h|\to 0} 2z+h = 2z$

We take advantage of $\mathbb{C}$ being a field.

In the limit, we could approach along x-axis, i.e.

$$h = \Delta x + i\Delta y, \quad \Delta y = 0$$

$$f(x+iy) = (x+iy)^2 = x^2 + 2ixy - y^2$$

$$\lim_{h=\Delta x \to 0} \frac{f(z+h)-f(z)}{h} = \frac{\partial}{\partial x} f = 2x + 2iy$$

along y axis

$$\lim_{h=i\Delta y \to 0} \frac{f(z+h)-f(z)}{h} = -i \frac{\partial}{\partial y} f = (-i)(-2y+2ix) = 2x+2iy$$

coming from here

Must have the equality. Otherwise the limit does not exist!

Thm (Cauchy-Riemann eqn)

Let $f: \mathbb{C} \to \mathbb{C}$ be a holomorphic fcn. on an open subset $U \subset \mathbb{C}$. Considering $f = u + iv$ as a real differential fcn on $\mathbb{R}^2$, then

$$\frac{\partial}{\partial x} u = \frac{\partial}{\partial y} v, \quad \frac{\partial}{\partial x} v = -\frac{\partial}{\partial y} u$$

Cor: Let f be a non-constant hol. fcn. $f: \mathbb{R}^2 \to \mathbb{R}^2$ is orientation preserving.

Pf: $J(f) = \begin{pmatrix} \frac{\partial}{\partial x} u & \frac{\partial}{\partial y} u \\ \frac{\partial}{\partial x} v & \frac{\partial}{\partial y} v \end{pmatrix}$ $\quad \det J(f) = \frac{\partial}{\partial x} u \frac{\partial}{\partial y} v - \frac{\partial}{\partial y} u \frac{\partial}{\partial x} v$

$$= \left(\frac{\partial}{\partial x} u\right)^2 + \left(\frac{\partial}{\partial x} v\right)^2 > 0$$

Jacobian matrix: a local approximation of f as a linear function.

For a linear fcn:

counterclockwise
↓
counterclockwise orientation preserving

Thm (open mapping thm)

A non-constant hol. fcn. f is open.

$$\left(\begin{array}{l} U \subset \mathbb{C} \text{ open} \\ \Rightarrow f(U) \subset \mathbb{C} \text{ is open} \end{array} \right)$$

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path integral

Given a path $\gamma : [a,b] \to \mathbb{C}$

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

(treat it like real integration)

Thm: Suppose $\gamma, \eta : [a,b] \to U \subseteq \mathbb{C}$ are related by a continuous deformation of paths.

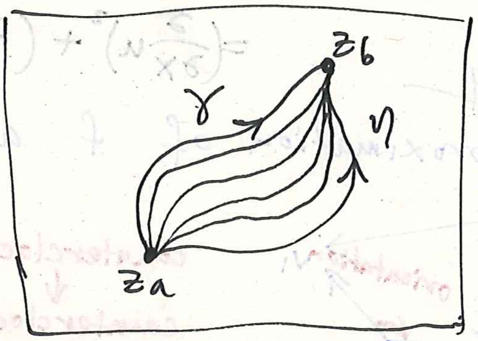
$$\left(\begin{array}{ll} H : [a,b]\times[0,1] \to U & \text{say } \gamma(a)=\eta(a)=z_a, \\ H(s,0)=\gamma(s), & \quad\gamma(b)=\eta(b)=z_b \\ H(s,1)=\eta(s). & H(a,t)=z_a \\ & H(b,t)=z_b \end{array} \right)$$

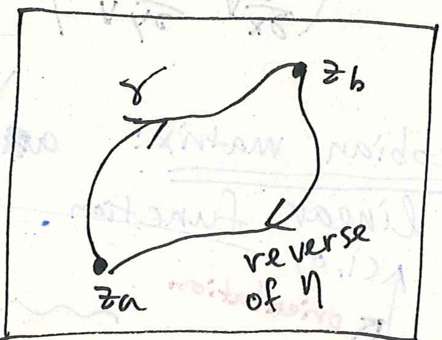
then

$$\int_\gamma f(z)\,dz = \int_\eta f(z)\,dz$$

(proof omitted)

proof can be done using Stoke's thm.





exercise

$$\int_\gamma f dz - \int_\eta f dz = \oint f dz = \ldots\ldots$$