<u>Defn</u>: A $y_0$-labeled map is a pair $(f, L)$ where $f: X \longrightarrow Y$ is a degree $d$ map of R.S. and $L: f^{-1}(y_0) \longrightarrow \{1, \cdots, d\}$ is a bijection.

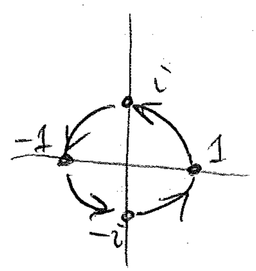
($y_0$ cannot be branch pt)

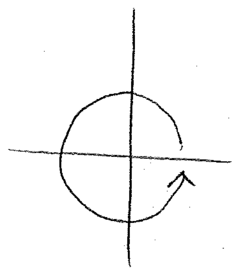
An isom. of $y_0$-labeled maps $(f_1, L_1)$, $(f_2, L_2)$ consists of an isom $\phi: X_1 \to X_2$ s.t.

$$f_2 \circ \phi = f_1 \qquad L_2 \circ \phi = L_1$$

<u>Ex</u>: $f: \mathbb{C} \longrightarrow \mathbb{C}$
$\qquad z \longmapsto z^4$

$y_0 = 1$, $\quad f^{-1}(y_0) = \{1, -1, i, -i\}$



$\gamma$: ~~the~~ loop of unit circle (counter clockwise)

$\sigma_\gamma: 1 \mapsto i \mapsto -1 \mapsto -i$

We can fix a labeling
$$\begin{matrix} \{1, & i, & -1, & -i\} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \{1, & 2, & 3, & 4\} \end{matrix}$$

Then $\sigma_\gamma = (1, 2\ 3\ 4) \in S_4$

Easy to check $\sigma_{\gamma * \eta} = \sigma_\gamma \circ \sigma_\eta$

A $y_0$-labeled map $(f: X \to Y, L)$ gives rise to a group homo

$$\Phi : \pi_1(Y \setminus B, y_0) \longrightarrow S_d$$
$$\gamma \longmapsto \sigma_\gamma$$

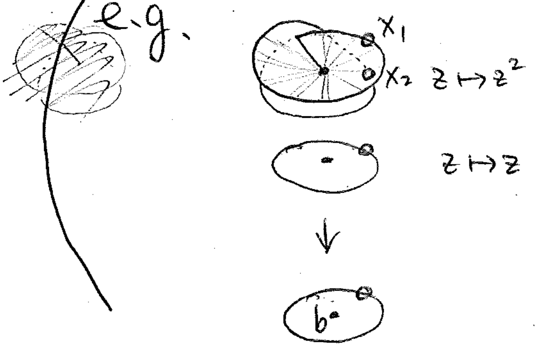
Called <u>monodromy representation</u>.

<u>Def</u>: A permutation whose cycle decomposition consists of disjoint cycles of length $\{l_1, \cdots, l_k\}$ is said to have <u>cycle type</u> $\{l_1, \cdots, l_k\}$

<u>Example</u>: $(12)(45)(368) \in S_8$ has cycle type $\{3,2,2,1\}$
("7" is fixed)

$y_0$-labeled map $f: X \to Y$. Suppose $f$ has ramification profile $\lambda = \{k_1, \cdots, k_l\}$ at a branch point $b \in Y$ and $\rho$ is a simple loop around $b$. Then $\Phi(\rho)$ has cycle type $\lambda$.

( e.g.




$\Phi(\rho)$ permutes $x_1, x_2$ )

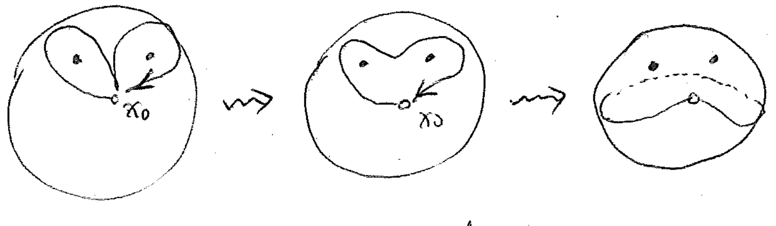
<u>Def</u>: Let $Y$ be a conn. R.S of genus $g$, and $y_0, b_1, \cdots, b_n \in Y$. Let $\lambda_1, \cdots, \lambda_n$ be partitions of a pos. int. $d$.

A <u>monodromy representation</u> of type $(g, d, \lambda_1, \cdots, \lambda_n)$ is a group homo. $\Phi : \pi_1(Y \setminus \{b_1, \cdots, b_n\}, y_0) \to S_d$ s.t. if $\rho_k$ is the homotopy class of a small loop around $b_k$, the permutation $\Phi(\rho_k)$ has cycle type $\lambda_k$.

If in addition the subgp $\mathrm{Im}\,\Phi \le S_d$ acts transitively on the set $\{1, 2, \cdots, d\}$, we say $\Phi$ is a <u>connected</u> monodromy repr.

a deg $d$ $y_0$-labeled map $\Longrightarrow$ a monodromy repr.

<u>Ex</u>: $\pi_1(\mathbb{P}^1 \setminus \{b_1, \cdots, b_n\}) = \mathbb{F}_{n-1}$ free group with $n-1$ generators




$\rightsquigarrow$ contracted

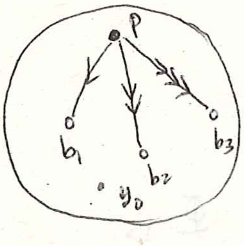
$\rho_i$ simple loop around $b_i$

$\rho_1 * \rho_2 * \cdots * \rho_n = e$

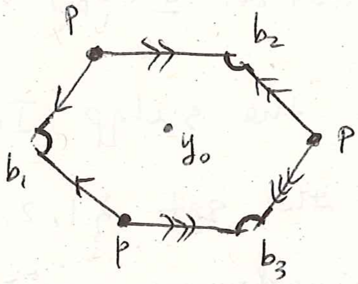
# Monodromy $\Longrightarrow$ Maps

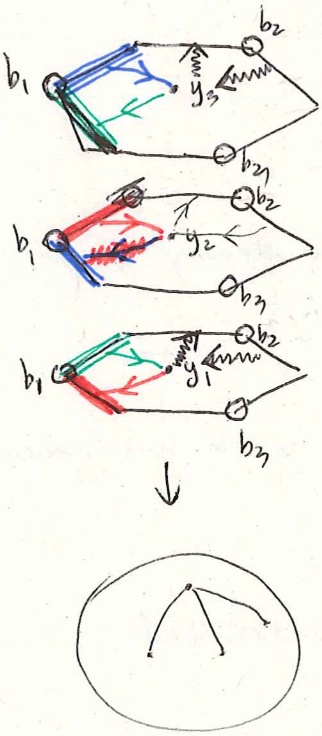
<u>Example</u>: Let $Y = \mathbb{P}^1$, $y_0, b_1, b_2, b_3 \in \mathbb{P}^1$.

Monodromy rep.: $\rho_1 \mapsto (1\,2\,3)$
$\rho_2 \mapsto (1\,3)$
$\rho_3 \mapsto (1\,2)$




flatten out




<u>$\rho_1$ determines:</u>

$y_1 \longrightarrow y_2 \longrightarrow y_3 \longrightarrow y_1$

<u>$\rho_2$ determines:</u>

$y_1 \rightsquigarrow y_3 \rightsquigarrow y_1$

$y_2 \longrightarrow y_2$.

<u>$\rho_3$: Exercise.</u>

Theorem: Given a monodromy repr. $\Phi$ of type $(g, d, \lambda_1, \cdots, \lambda_n)$ for any R.S. $Y$ of genus $g$ and $B = \{b_1, \cdots, b_n\} \in Y$, $\exists$ a $y_0$-labeled map of R.S. covering $Y$ with branch locus $B$, whose associated monodromy repr. is $\Phi$.

Such a map is unique up to isom. of $y_0$-labeled maps.

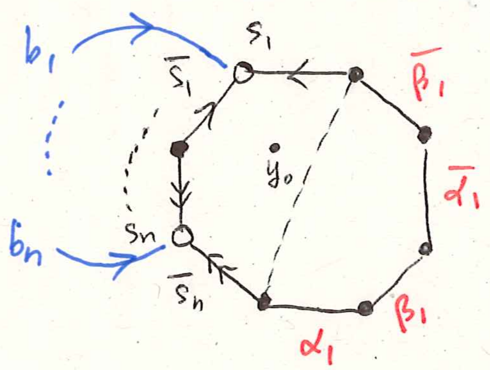
Pf: 1) Represent $Y$ as an identification polygon of type $\alpha_1 \beta_1 \bar{\alpha}_1 \bar{\beta}_1 \cdots \alpha_g \beta_g \bar{\alpha}_g \bar{\beta}_g$.

(connect sum of $g$ tori)

$Y$ can also be represented as

$$s_1 \bar{s}_1 \, s_2 \bar{s}_2 \cdots s_n \bar{s}_n \, \alpha_1 \beta_1 \cdots \bar{\alpha}_g \bar{\beta}_g$$

$(S^2 \# S^2 \# \cdots S^2 \# T^{\# g})$




remove vertices at $s_1 \bar{s}_1$, $s_2 \bar{s}_2$, $\cdots$

~~open~~

$= P = Y \setminus \{b_1, \cdots, b_n\}$

2) Take $\underbrace{P \amalg P \amalg \cdots \amalg P}_{d = \text{degree of the cover}}$

and glue edges

1st $P$ : $s_{1,1} \bar{s}_{1,1} \cdots \bar{s}_{n,1} s_{n,1} \alpha_{1,1} \beta_{1,1} \bar{\alpha}_{1,1} \bar{\beta}_{1,1} \cdots$

2nd $P$ : $s_{1,2} \bar{s}_{1,2} \cdots$

$\vdots$ $P$ : $\vdots$

$d$th $P$ : $s_{1,d} \bar{s}_{1,d} \cdots$

$\pi_1(Y \setminus \{b_1, \cdots, b_n\})$ is generated by

$\{\alpha_i, \beta_i, \rho_j\}$ where $\rho_j$ wraps around $b_j$

~~$\Phi_i: \{\rho_j\} \to \{\rho_j\}$ satisfying group relations in $\pi_1$.~~

~~$\{\beta_i\} \to \{\beta_i\}$~~

~~$\{\alpha_i\} \to \{\alpha_i\}$~~

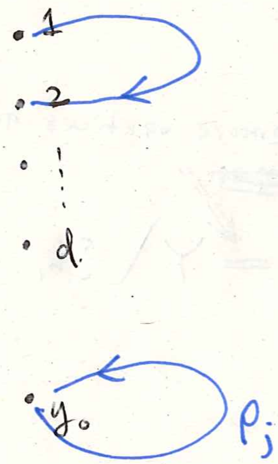
~~$\rho_j, \alpha_i, \beta_i$ act on the set $\{1, \cdots, d\}$~~

~~Gluing~~

~~$s_{j,k} \sim \overline{s}_{j, \Phi(\rho_j)k}$~~

$\Phi(\alpha_i)$, $\Phi(\beta_i)$, $\Phi(\rho_j)$ are actions on the set $\{1, 2, \cdots, d\}$ defined by lifting the loop ~~to~~ to a fiber of $y_0$ and look for the end point.

E.g.




$\Longrightarrow \Phi(\rho_j)(1) = 2$

Gluing of P's :

$s_{j,k} \sim \overline{s}_{j, \Phi(\rho_j)k}$

$\overline{\alpha}_{i,k} \sim \alpha_{i, \Phi(\beta_i)k}$

$\beta_{i,k} \sim \overline{\beta}_{i, \Phi(\alpha_i)k}$

The rest : Refer to the textbook

We are left to make some formal argument about uniqueness.

# Monodromy repr. and Hurwitz #



<u>Thm</u>: Let $M$ be the set of conn. monodromy repr. of type $(g, d, \lambda_1, \cdots, \lambda_n)$. Then

$$H_{h \xrightarrow{d} g}(\lambda_1, \cdots, \lambda_n) = \frac{|M|}{d!}$$

not the # of automorphism of the cover!

Pf: $m_f$ = # of isom classes of $y_0$ labeled map $f$

= # of distinct monodromy repr. arising from $f$ by diff labelling

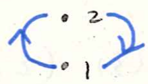
$$= \frac{d!}{|\mathrm{Aut}(f)|}$$

permutation of $d$ points

$$H_{h \xrightarrow{d} g}(\lambda_1, \cdots, \lambda_n) = \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|} = \sum_{[f]} \frac{m_f}{d!} = \frac{1}{d!} \sum m_f = \frac{|M|}{d!}$$

□

E.g. hyperell. cover




No matter how to label the two pts, monodromy repr. are the same due to the automorphism of the cover.

Thus, automorphism of the map is already dealt with in $|M|$.

# Examples :



- $H_{g\xrightarrow{2}0}((2)^{2g+2})$ in terms of monod. repr.

$Y = \mathbb{P}^1$, $B = \{b_1, \ldots, b_{2g+2}\}$

$\pi_1(Y \setminus B, y_0) = \langle \rho_1, \cdots, \rho_{2g+2} \rangle / \rho_1 \cdots \rho_{2g+2} = e$

$\Phi(\rho_i) = (1\,2)$ is the only choice

Only one monod. repr $\Rightarrow H_{g\xrightarrow{2}0}((2)^{2g+2}) = \frac{1}{2}$

- $H_{0\xrightarrow{3}0}((3), (2,1)^2)$

$\Phi(\rho_1) \to (1\,2\,3)$, $(1\,3\,2)$

$\Phi(\rho_2) \to (1\,2)$, $(2\,3)$, $(1\,3)$

$\Phi(\rho_3) = \Phi(\rho_1)^{-1} \circ \Phi(\rho_2)^{-1}$

So $H_{0\xrightarrow{3}0}((3), (2,1)^2) = \dfrac{2 \times 3}{3!} = 1$

- $H_{0\xrightarrow{3}0}((2,1)^4)$ and $H^{\bullet}_{0\xrightarrow{3}0}((2,1)^4)$

Disconnected:

$\left.\begin{array}{l} \rho_1 \\ \rho_2 \\ \rho_3 \end{array}\right\}$ free to choose 2-cycles.

$\Phi(\rho_4) = \Phi(\rho_1)^{-1} \circ \Phi(\rho_2)^{-1} \circ \Phi(\rho_3)^{-1}$

$H^{\bullet}_{0\xrightarrow{3}0}((2,1)^4) = \dfrac{3 \times 3 \times 3 \times 1}{3!} = \dfrac{9}{2}.$

## Disconnected monodromy repr:

$\Phi(\rho_1) = \Phi(\rho_2) = \Phi(\rho_3)$ (otherwise the action is transitive)

$$H_{0\xrightarrow{3}0}((2,1)^4) = \frac{3^3-3}{3!} = 4$$

- $H^{\bullet}_{1\xrightarrow{d}1}(\phi)$

$$\pi_1(T) = \underset{\substack{\uparrow \\ a}}{\mathbb{Z}} \oplus \underset{\substack{\uparrow \\ b}}{\mathbb{Z}} \xrightarrow{\Phi} S_d$$

$\Phi(a)$ and $\Phi(b)$ commutes $\Rightarrow$ $\Phi(b)$ in the centralizer of $\Phi(a)$

## Centralizer and conjugacy class:

$G \circlearrowright G$ by conjugation
($h$ acts on $g$ by $hgh^{-1}$)

Pick any $g \in G$

Orbit: $\underbrace{\{hgh^{-1}\}}_{=C_g} \subset G$, conjugacy class of $g$.

Stabilizer: $\underbrace{\{h \mid hgh^{-1} = h\}}_{=\xi(g)} < G$, centralizer of $\Phi(a)$

$$|G| = |\xi(g)|\,|C_g|$$

So $H^{\bullet}_{1\xrightarrow{d}1}(\phi) = \frac{1}{d!}\sum_{\sigma\in S_d} \frac{d!}{|C_\sigma|} = \sum_{\sigma \in S_d} \frac{1}{|C_\sigma|}$ = # of conjugacy classes in $S_d$ (partitions of $d$)

- $H_{0\xrightarrow{4}0}((3),(2,2)^2) = 0.$

$\rho_1 \to (3)$ cycle (fix one element)
$\rho_2 \to (2)(2)$ cycle
$\rho_3 \to (2)(2)$ cycle.

1 2 3 4

Say 1 is fixed by $\Phi(\rho_1)$.

$1 \xrightarrow{\rho_2} i \xrightarrow{\rho_3} 1$

$\Rightarrow$ $\Phi(\rho_2)$ and $\Phi(\rho_3)$ both contain cycle $(1\ i)$, contradiction. □

- $H^{\bullet}_{2g-1 \xrightarrow{2} g}(\phi)$.

$\pi_1(Y) = \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g \rangle / (\alpha_1\beta_1 \cdots \alpha_g^{-1}\beta_g^{-1})$

$S_2$ is abelian $\Rightarrow$ no restriction in choosing $\Phi(\alpha_i), \Phi(\beta_i)$

$H^{\bullet}_{2g-1 \xrightarrow{2} g}(\phi) = 2^{2g}/2! = 2^{2g-1}$

Disconn. monod. repr: $\Phi(\alpha_i) = \Phi(\beta_i) = e$

$H_{2g-1 \xrightarrow{2} g}(\phi)\ \dfrac{2^{2g}-1}{2!}$