Cauchy integral formula

Thm Let γ be a small loop around $z \in \mathbb{C}$ and $f(w)$ a holomorphic fcn in a nbh. U of γ. Then

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z}\,dw$$

Sketch of proof: $\oint \frac{1}{w-z}\,dw = 2\pi i$

$$\frac{1}{2\pi i}\oint \frac{f(w)}{w-z}\,dw = f(z)$$

So we just need to prove:

$$\frac{1}{2\pi i}\oint \frac{f(w)-f(z)}{w-z}\,dw = 0$$

argue this guy is holomorphic ---

Cor: holomorphic fcn. are analytic

(Infinitely differentiable + Taylor expansion converges in a nbh of given pt.)

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z_0-(z-z_0)}\,dw = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z_0}\,\frac{1}{1-\frac{z-z_0}{w-z_0}}\,dw$$

$$= \frac{1}{2\pi i}\oint \frac{f(w)}{w-z_0}\left[1+\left(\frac{z-z_0}{w-z_0}\right)+\left(\frac{z-z_0}{w-z_0}\right)^2+\dots\right]dw$$

have nothing to do with ~~a~~ a

Each summand: $\underbrace{\left(\frac{1}{2\pi i}\oint \frac{f(w)}{(w-z_0)^{i+1}}\,dw\right)}_{\text{a number}}\underbrace{(z-z_0)^i}_{\text{a monomial}}$

a number
↓
coefficient of Taylor expansion

Def: a complex fcn f has a pole of order n at the point $z_0 \in \mathbb{C}$ if $(z-z_0)^n f(z)$ is holomorphic at z_0 but $(z-z_0)^{n-1} f(z)$ isn't.

Cor: If f has a pole of order n at z_0, then it admits a Laurent expansion in a nbh. of z_0:

$$f(z) = \sum_{k=-n}^{\infty} a_k (z-z_0)^k$$

($\forall z_0 \in U \subset \mathbb{C}$, f either hol. or has a pole at $z_0 \Rightarrow f$ is a meromorphic fcn)

Def: Let f have a pole at z_0. The *residue* of f at z_0 is the $k=-1$ coefficient in the Laurent expansion of f at z_0. denoted by $\mathrm{Res}_{z=z_0} f(z)$

Exercise 1.3.3

f has poles at $z_1, \cdots, z_m$ and holomorphic on other pts.

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}_{z=z_j} f(z)$$

Ex 1.3.2
If f has a pole of order 1 at z_0, then the residue of f at z_0 =

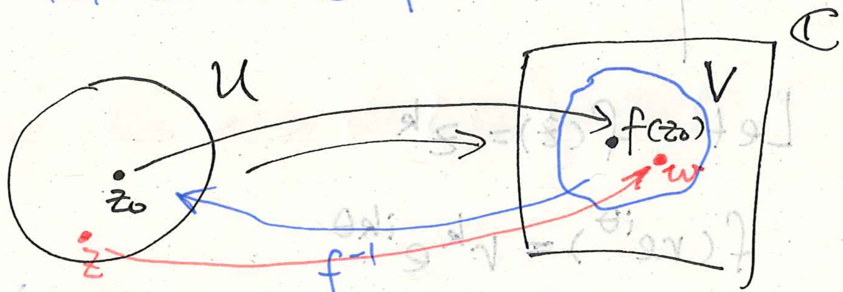
$$\mathrm{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z-z_0) f(z)$$

Inverse function

Thm Let $f: U \to \mathbb{C}$ be a holomorphic fcn. and $z_0 \in U$ s.t. $f'(z_0) \neq 0$. Then $\exists$ a nbh. V of $f(z_0)$, and a hol. fcn. $g: V \to \mathbb{C}$ s.t. $z_0 \in g(V)$, and $\forall z \in g(V)$, $g \circ f = id$.

Could be proven by other methods. But the one in the textbook is pretty cool!

pf:



exists at least as set fcn/differentiable fcn in a nbh.

$$U \underset{f^{-1}}{\overset{f}{\rightleftarrows}} V$$

$\forall w \in V$,

$$g(w) := \frac{1}{2\pi i}\oint_\gamma \frac{\zeta f'(\zeta)}{f(\zeta)-w}\, d\zeta$$

$$= \mathrm{Res}_{\zeta=z}\left(\frac{\zeta f'(\zeta)}{f(\zeta)-w}\right)$$

because it's holomorphic away from $\zeta = z$.

$\lim_{\zeta \to z} (\zeta - z)\frac{\zeta f'(\zeta)}{f(\zeta)-w}$ exists. The limit is

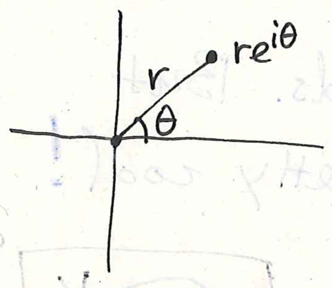
$$\frac{z\cancel{f'(z)}}{\cancel{f'(z)}} = z\,!$$

So ~~g~~ $g(w) = f^{-1}(w)$

$g(w)$ is holomorphic by definition.

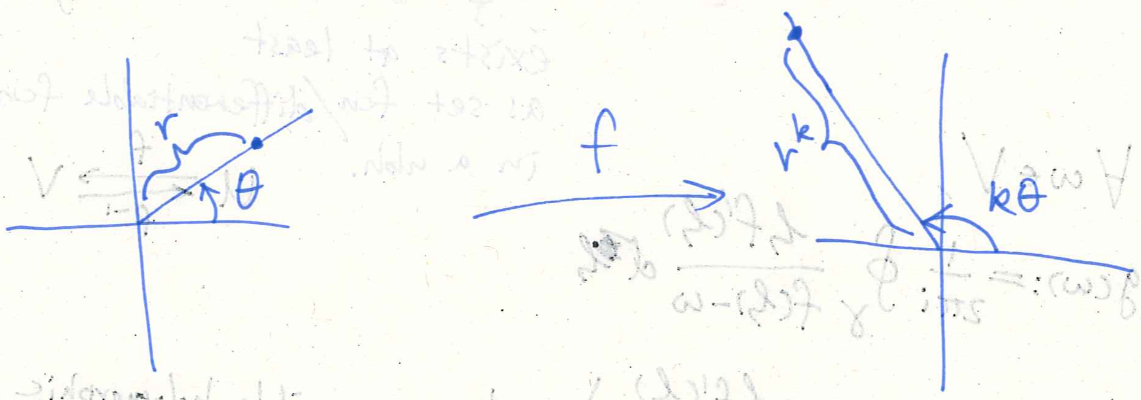
The k-th root "function"

A complex number z can be written as $z = r \cdot e^{i\theta}$

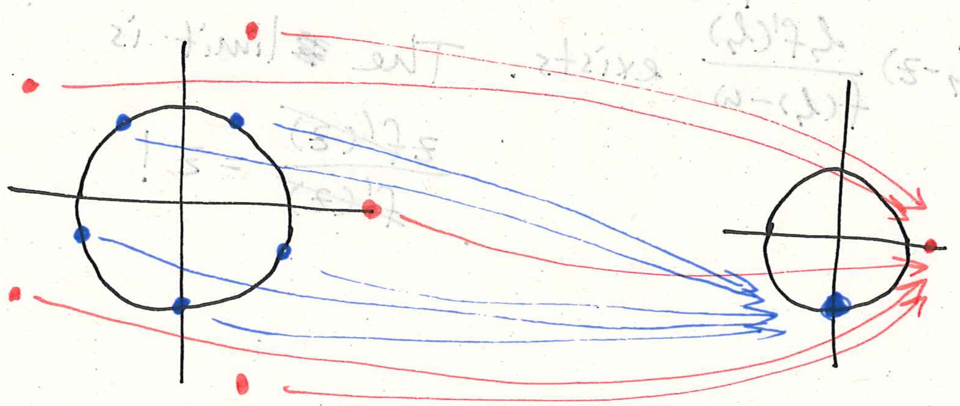


Let $f(z) = z^k$

$f(re^{i\theta}) = r^k e^{ik\theta}$



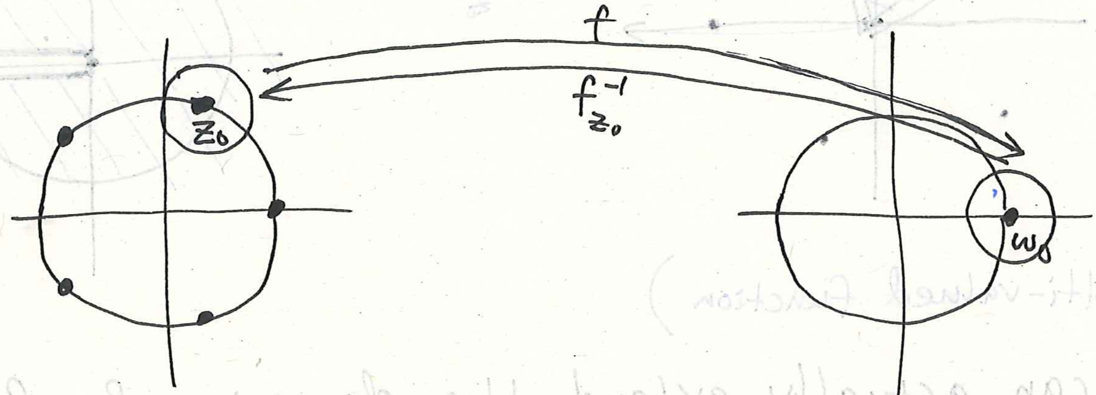
Say $k=5$



f does not have an inverse.

f is $k:1$ - function except at $z=0$.

But when $z\neq 0$, f is <u>locally</u> invertible at z.

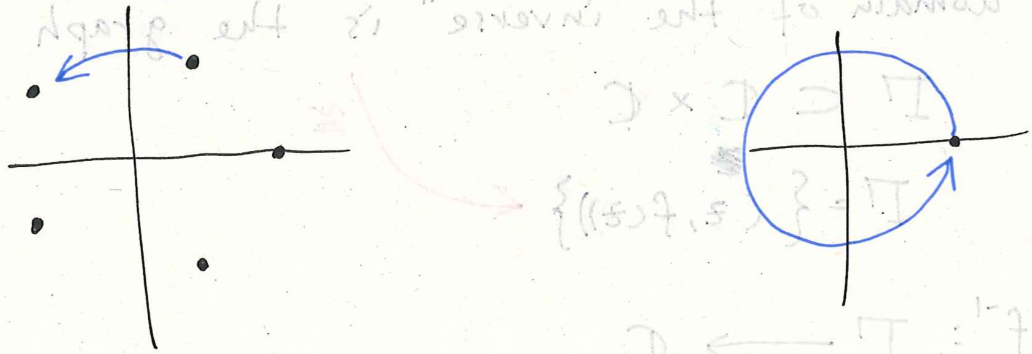


For a w_0, need to fix a z_0 such that $f(z_0)=w_0$. the local inverse is denoted (according to the textbook) by $f_{z_0}^{-1}$

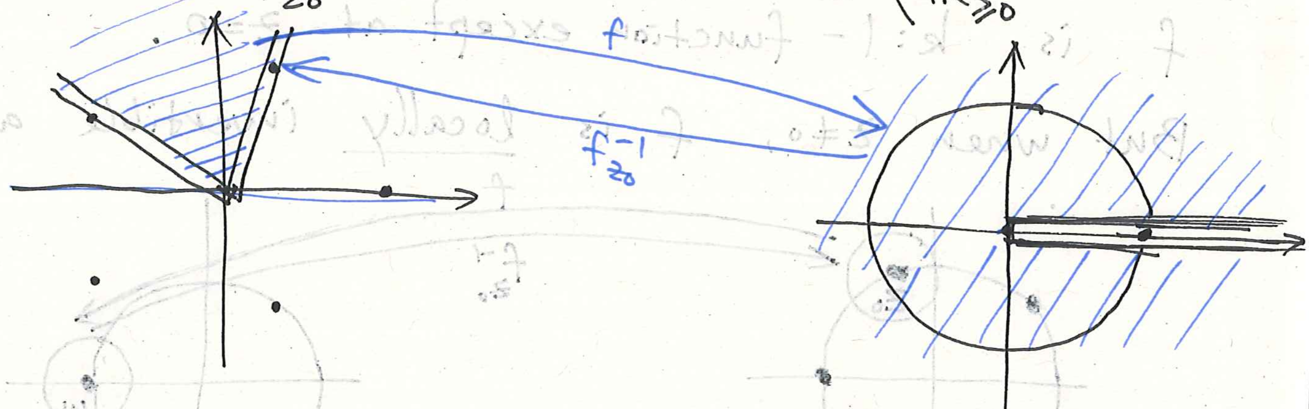
Question:

Let t goes (start at w_0) along unit circle counterclockwise and keep extending the local inversion continuously, what is the limit of $f_{z_0}^{-1}(t)$ when t comes back to w_0?

(Just imagine the target is a clock that runs k times as fast as the domain.)



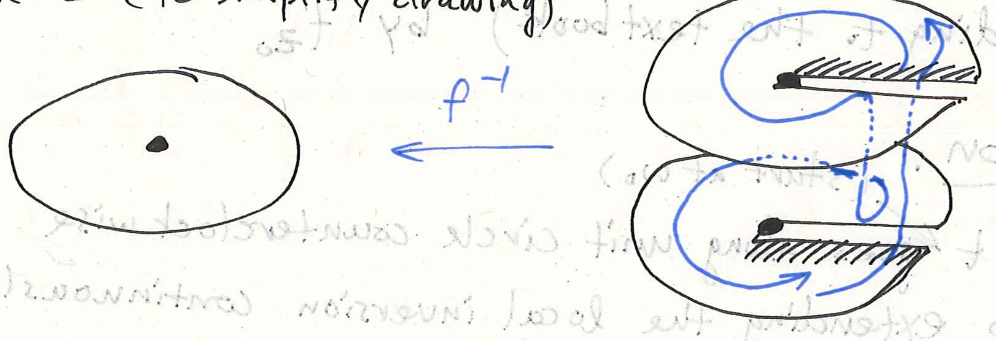
In fact, $f_{z_0}^{-1}$ is defined on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$



(Multi-valued function)

We can actually extend the domain of $f_{z_0}^{-1}$ further beyond $\mathbb{C}$ and make it a spiral

Say $k=2$ (to simplify drawing)



More rigorously,

$$f : \mathbb{C} \longrightarrow \mathbb{C}$$

the domain of the "inverse" is the graph

$$\Gamma \subset \mathbb{C} \times \mathbb{C}$$

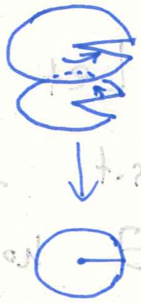
$\cong$

$$\Gamma = \{ (z, f(z)) \}$$

$$f^{-1} : \Gamma \longrightarrow \mathbb{C}$$

$$\begin{tikzcd} & \Gamma \arrow[d, "\pi_2"] \\ \mathbb{C} \arrow[ur] \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

$$z \longmapsto (z, f(z)) \longmapsto f(z)$$



Replace target by a covering
now inverse can be defined.

Γ : the Riemann surface of the k-th root

The covering $\pi_2 : \Gamma \to \mathbb{C}$ (projection to 2nd factor) is a ramified cover. It's ramified over $0 \in \mathbb{C}$.

($\Gamma \xrightarrow{\pi_2} \mathbb{C}$ is not a covering space!
$\Gamma \setminus \{(0,0)\} \xrightarrow{\pi_2} \mathbb{C} \setminus \{0\}$ is a covering space.)

Ex 1.4.4 Let $g(w)$ be a holomorphic fcn. Show that for any w_0 s.t. $g(w_0) \neq 0$, there exists a nbh. of w_0, and k distinct choices (branches) for a hol. map $\tilde{g}$ s.t. $\tilde{g}^k(w) = g(w)$

Lem: Let $w = f(z)$ be a hol. fcn. and $z_0 \in \mathbb{C}$ s.t $f^{(i)}(z_0) = 0$ $i = 1, \cdots, k-1$, $f^{(k)}(z_0) \neq 0$.
Then $\exists$ hol. ~~fcn.~~ $\tilde{z}(z)$, $\tilde{w}(w)$ s.t.
Change of variable.
under variables $\tilde{z}, \tilde{w}$, f becomes $\tilde{w} = \tilde{z}^k$

(or precisely, $\tilde{w}(f(z)) = \tilde{z}^k(z)$)

Sketch of Pf.

Taylor expansion

$$f(z) - f(z_0) = \sum_{n=k}^{\infty} a_n (z - z_0)^n, \quad a_k \neq 0$$

$$g(z) = \sum_{n=k}^{\infty} a_n (z - z_0)^{n-k}, \quad g(z_0) \neq 0, \cancel{\exists \text{ a branch}}$$

$\Rightarrow$ It admits a branch of the k-th root around z_0
(locally near z_0, $\sqrt[k]{g(z)}$ is defined and hol.)

$\tilde{z} = (z - z_0)\sqrt[k]{g(z)}$ locally near $z = z_0$.

Check it is invertible.

$\tilde{w} = w - f(z_0)$.

Manifold theory

Def: A topological space X is called a (smooth) manifold if

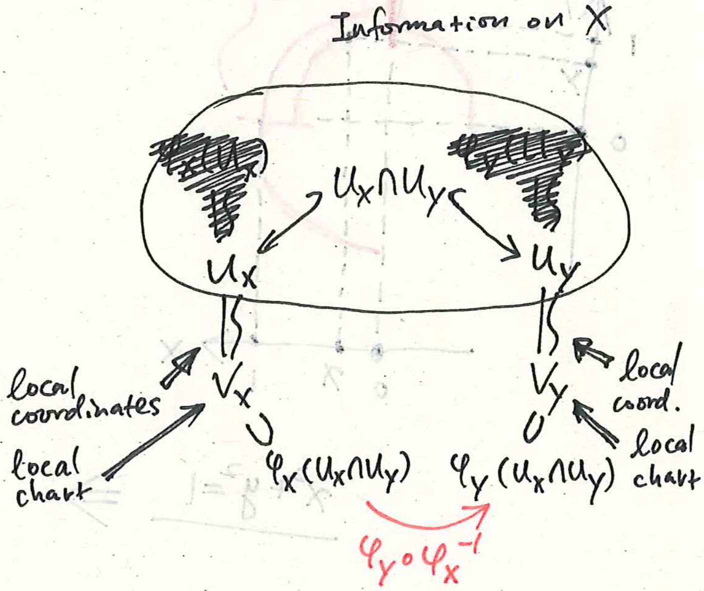
1. X is Hausdorff
2. $\forall\, x \in X$, $\exists$ a nbh. $U_x \subset X$ of x, and a homeomorphism $\varphi_x : U_x \to V_x$ where $V_x \subset (\mathbb{R}^n)$ is an open set.
3. $\forall\, U_x, U_y$, s.t. $U_x \cap U_y \neq \phi$, the transition fcn.

$$T_{y,x} : \varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \longrightarrow \varphi_y(U_x \cap U_y)$$

is (smooth).
(or infinitely differentiable)

(smooth) $\to$ • C^k differential
• complex analytic fcn
...

$(\mathbb{R}^n) \to$ • $\mathbb{R}^n$
• $\mathbb{C}^n$
...



The collectn $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is called an atlas.