Orientability:

$\exists$ an atlas s.t. all transition functions are orientation-preserving.

A surface is non-orientable iff it contains a Möbius strip.

$\mathbb{P}^2(\mathbb{R})^{\# m}$ are non-orientable

S^2, $T^{\# g}$ are orientable.

Manifolds as level sets

A way to construct manifold is to make it the zero set of a function.

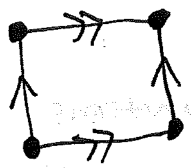
Thm (Implicit function theorem) (holomorphic)

Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ $(\mathbb{C}^n) \rightarrow (\mathbb{C}^m)$ be a smooth function, and $x \in \mathbb{R}^n$ $(\mathbb{C}^n)$ s.t. the differential $dF(x)$ is a surjective linear function. Say $F(x) = a$.

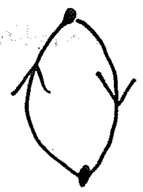
Then there exists:

- $V_x \subseteq \mathbb{R}^n$ an open nbh. of x
- $U_x \subseteq \mathbb{R}^{n-m}$ an open set
- $f_x: U_x \rightarrow \mathbb{R}^m$ a smooth fcn holomorphic

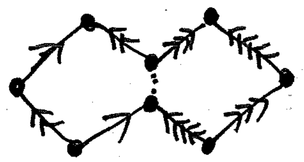
Ex:

•  $= T$

$\chi(T) = 1 - 2 + 1 = 0$

•  $= \mathbb{P}^2(\mathbb{R})$

$\chi(\mathbb{P}^2(\mathbb{R})) = 1 - 1 + 1 = 1$

• $T \# T =$ 

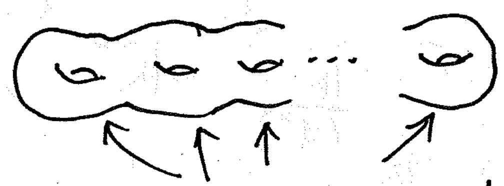
$\chi(T \# T) = 1 - 4 + 1 = -2$

Ex: What is $\chi(T^{\# g})$?

For orientable surfaces $X \cong T^{\# g}$,

g : genus of X

"number of holes"



add up to g holes.

Ex: Klein bottle $\cong \mathbb{P}^2(\mathbb{R}) \# \mathbb{P}^2(\mathbb{R})$

$T \# \mathbb{P}^2(\mathbb{R}) \cong \mathbb{P}^2(\mathbb{R}) \# \mathbb{P}^2(\mathbb{R}) \# \mathbb{P}^2(\mathbb{R})$

Euler characteristic and Orientability

Def : A good graph on a surface S is a graph Γ on S such that :

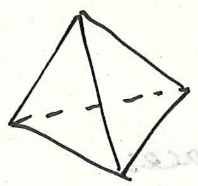
1. $S \setminus \Gamma$ is homeomorphic to a disjoint union of open disks.
2. wherever two edges cross, there is a vertex
3. no edge ends without a vertex

☆ Euler characteristic

$$\chi(S) = |V_\Gamma| - |E_\Gamma| + |F_\Gamma|$$

\# of vertices   \# edges   \# faces.

$\chi(S)$ is independent of the choice of good graphs.

Ex:  $= X$

$$\chi(X) = 4 - 6 + 4 = 2$$

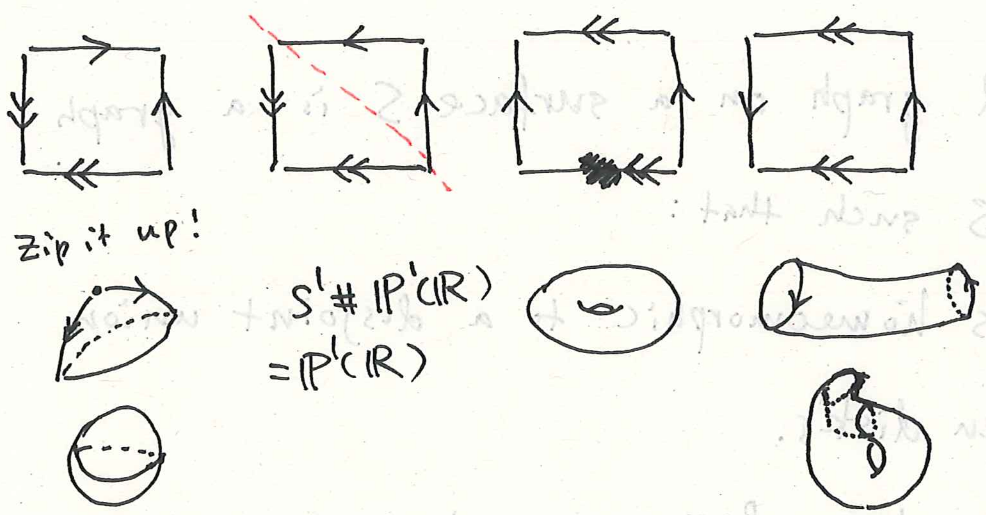
$X \cong S^2$, thus $\chi(S^2) = 2$

Ex: $w_1 = a\bar{a}b\bar{b}$

$w_2 = aab\bar{b}$

$w_3 = ab\bar{a}\bar{b}$

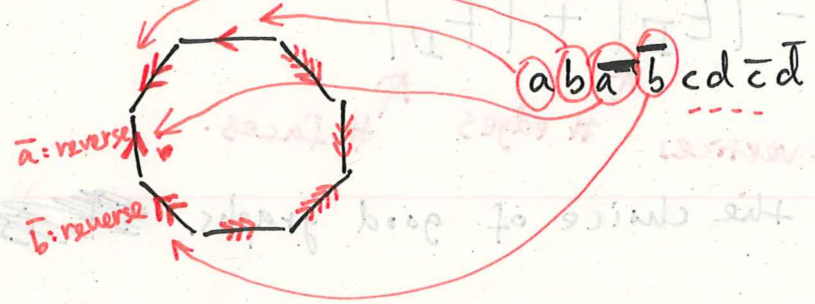
$w_4 = aba\bar{b}$



In general, given an identification polygon w with $2n$ sides, consider a regular $2n$-gon.

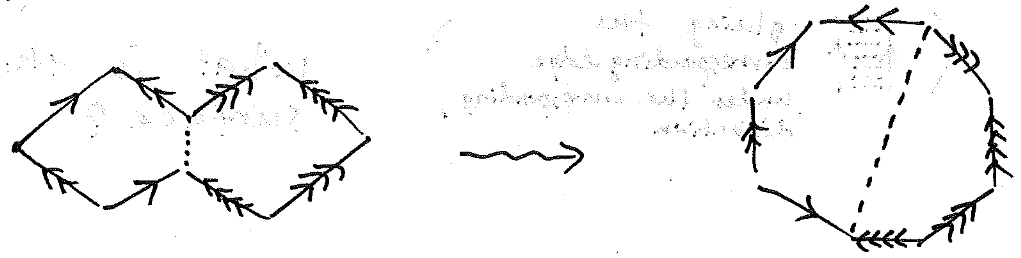
~~counterclockwise~~

Label its sides counterclockwise.



Exercise: If S_1, S_2 are connected compact surfaces represented by w_1, w_2. Show $S_1 \# S_2$ can be represented by $w_1 w_2$. We assume the two words use different sets of alphabets. A_1 and A_2.

- How to express $T \# T$?



-

A systematic way to denote surfaces after gluing

Notations:
- a set A of n letters : an alphabet
- $A \cup \bar{A}$ repeating each letter a second time : a double alphabet.
- Each pair $a, \bar{a}$: a pair of twin letters.

Ex. An alphabet of two letters :

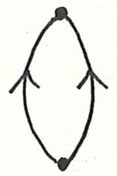
$$A = \{a, b\}$$

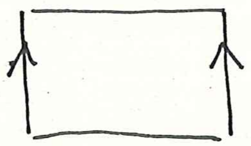
Doubled alphabet : $A \cup \bar{A} = \{a, b, \bar{a}, \bar{b}\}$

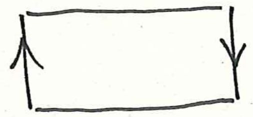
Twin pairs : $\{a, \bar{a}\}$, $\{b, \bar{b}\}$

Def: An identification polygon with $2n$ sides is a word w constructed from a double n-letter alphabet such that, for each pair of twin letters, w contains exactly two letters from the pair.
In particular, the word w must have $2n$ letters.

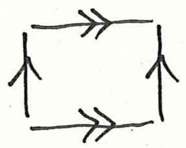
Identification of polygons

-  ( gluing the corresponding edge under the corresponding direction) what is this surface?

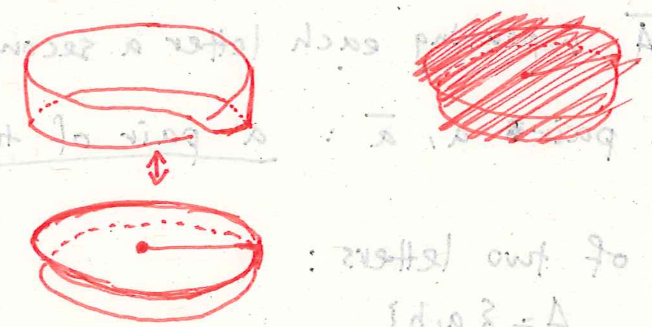
-  what is this surface?

-  what is this surface?

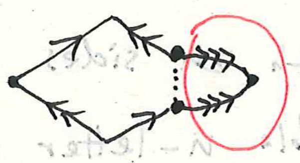
- ~~what is this~~

-  ?

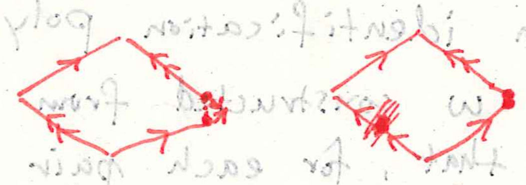
-  ? (can we see this in 3d space ?)



- How to express $T\#S^2$



this part can shrink arbitrarily.



(we won't do it rigorously.)

To make it rigorous, need to prove that the ~~diffeomorphic~~ homeomorphic type of $S_1 \# S_2$ is independent of the choice of small discs removed from S_1 and S_2.

(Ex 2.4.1)

Thm (Classification of Compact Surface)

Any connected compact surface is homeomorphic to exactly one of the following:

- S_2
- $T^{\# g} = \underbrace{T \# T \# \cdots \# T}_{g}$
- $\mathbb{P}^2(\mathbb{R})^{\# m} = \underbrace{\mathbb{P}^2(\mathbb{R}) \# \cdots \# \mathbb{P}^2(\mathbb{R})}_{m}$

orientable mfds.

non-orientable mfds

Can be distinguished by orientability and Euler characteristic.

Charts:

$\mathbb{P}^2(\mathbb{R})$ open because $\{[x_0 : x_1 : x_2] \mid x_0 \neq 0\} = \{[1 : x_1 : x_2]\}$ and $\{(x_0, x_1, x_2) \mid x_0 \neq 0\}$ is open in $\mathbb{R}^3$

$U_1 = \{[1 : x_1 : x_2]\}$, $U_2 = \{[x_0 : 1 : x_2]\}$

$U_3 = \{[x_0 : x_1 : 1]\}$

$U_1 \cong \mathbb{R}^2$ because x_1, x_2 can be chosen arbitrarily

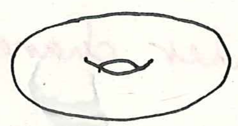
(more rigorously, show $\{(1, x_1, x_2) \in \mathbb{R}^3\}/\sim \;\cong \mathbb{R}^2$ is isomorphic to $\mathbb{R}^2$ under quotient topology. This is obvious because "$\sim$" is empty.)

One can replace $\mathbb{R}$ by $\mathbb{C}$!

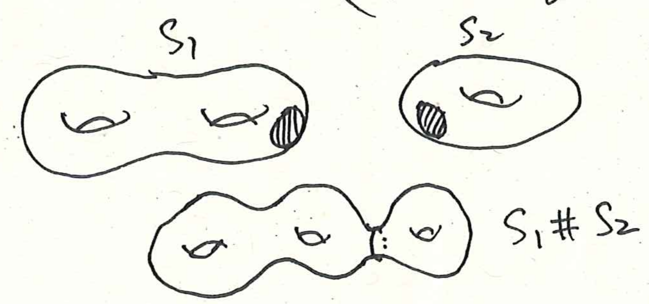
Ex $\mathbb{P}^1(\mathbb{C}) \cong S^2$

Compact surfaces

surfaces → a manifold of real dimension 2

Example: $\mathbb{P}^2(\mathbb{R})$, $\mathbb{P}^1(\mathbb{C})$, torus T 

Connected sum: ~~choose disc~~ (S_1, S_2 surfaces. Remove small discs on S_1, S_2 and glue together.)

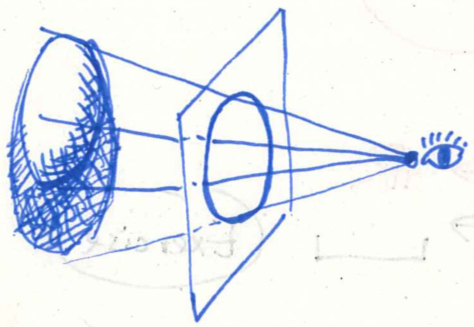


Projective spaces

- Project a 3-d object to a 2-d surface: (drawing)

"line of sight" = "point"

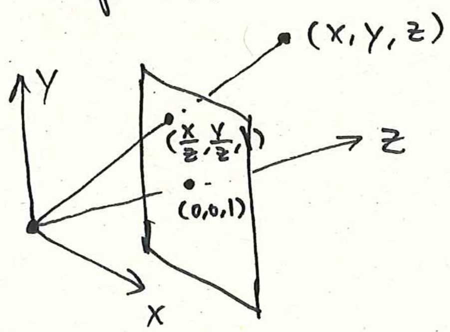
(I don't know what I'm drawing...) 3d 2d



Def The projective space $\mathbb{P}^n(\mathbb{R})$ is defined to be $\mathbb{R}^{n+1}\setminus\{0\} \big/ (x_0,\cdots,x_n) \sim (\lambda x_0,\cdots \lambda x_n)$, under quotient topology.

The tuple $[x_0 : x_1 : \cdots : x_n]$: homogeneous coordinate

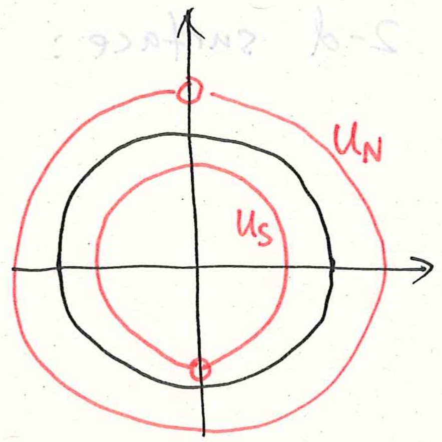
$[x_0 : x_1 : \cdots : x_n]$ and $[\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n]$ represent the same point.



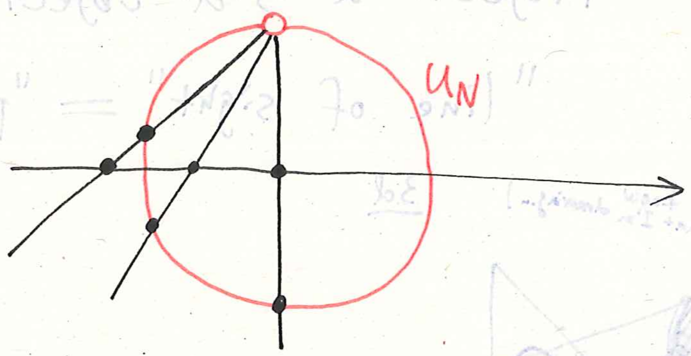
$[x : y : z] = [\frac{x}{z} : \frac{y}{z} : 1]$ if $z \neq 0$,

When $z = 0$, $[x : y : 0]$ should be thought of as points at "infinity"

Another atlas for S^1:



stereographic projection



$U_N \longrightarrow \mathbb{R}$

$(x,y) \longmapsto \_\_\_$ Exercise

$U_S \longrightarrow \mathbb{R}$

$(x,y) \longmapsto \_\_\_$ Exercise

$$\begin{tikzcd}
 & U_S \cap U_N \arrow[dl, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\supset"] & \\
U_S \arrow[d, phantom, sloped, "\cong"] & & U_N \arrow[d, phantom, sloped, "\cong"] \\
\mathbb{R} \arrow[d, phantom, sloped, "\supset"] & & \mathbb{R} \arrow[d, phantom, sloped, "\supset"] \\
\mathbb{R}\setminus\{0\} \arrow[rr] & & \mathbb{R}\setminus\{0\}
\end{tikzcd}$$

Exercise

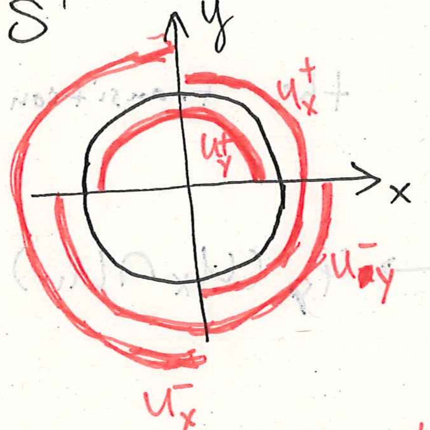
Def: Two atlases $\mathcal{A}=\{(U_\alpha,\varphi_\alpha)\}_\alpha$, $\mathcal{B}=\{(U_\beta,\varphi_\beta)\}_\beta$ for a topological space is called compatible, if their union $\mathcal{A}\cup\mathcal{B}$ is an atlas for X.

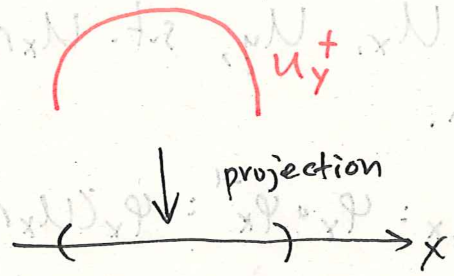
(for all α,β s.t. $U_\alpha\cap U_\beta\neq\emptyset$, the transition function $\varphi_\beta\circ\varphi_\alpha^{-1}$, $\varphi_\alpha\circ\varphi_\beta^{-1}$ are (smooth))

(Ex 2.1.1)

Examples:

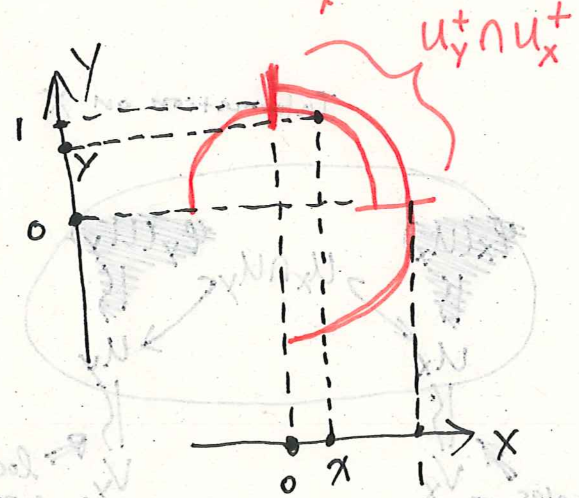
- S^1





$$\begin{array}{c} U_y^+ \\ \wr\big\downarrow \varphi_y^+ \\ (-1,1) \end{array}$$

similar for U_y^-, U_x^+, U_x^-



$x^2+y^2=1 \Rightarrow y=\sqrt{1-x^2}$

$$\begin{tikzcd}
 & U_y^+\cap U_x^+ \arrow[dl, phantom, "\subset"] \arrow[dr, phantom, "\supset"] & \\
U_y^+ \arrow[d, "\wr\ \varphi_y^+"] & & U_x^+ \arrow[d, "\wr\ \varphi_x^+"] \\
(-1,1) & & (-1,1) \\
(0,1) \arrow[u, phantom, "\cup"] \arrow[rr, "\varphi_x^+\circ(\varphi_y^+)^{-1}"] & & (0,1) \arrow[u, phantom, "\cup"] \\
x \arrow[rr, mapsto] & & \sqrt{1-x^2}
\end{tikzcd}$$

in x-coord

in y-coord

Manifold theory

Def: A topological space X is called a (smooth) manifold if

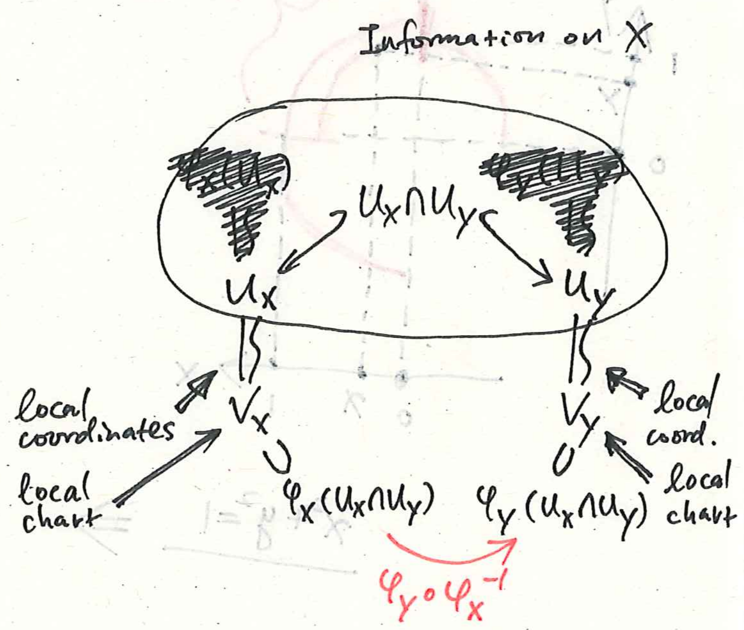
1. X is Hausdorff
2. $\forall\, x \in X$, $\exists$ a nbh. $U_x \subset X$ of x, and a homeomorphism $\varphi_x : U_x \to V_x$ where $V_x \subset (\mathbb{R}^n)$ is an open set.
3. $\forall\, U_x, U_y$, s.t. $U_x \cap U_y \neq \phi$, the transition fcn.

$$T_{y,x} : \varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \longrightarrow \varphi_y(U_x \cap U_y)$$

is (smooth).
(or infinitely differentiable)

(smooth) $\rightarrow$ • C^k differential
• complex analytic fcn
...

$(\mathbb{R}^n) \rightarrow$ • $\mathbb{R}^n$
• $\mathbb{C}^n$
...



The collectn $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is called an atlas.