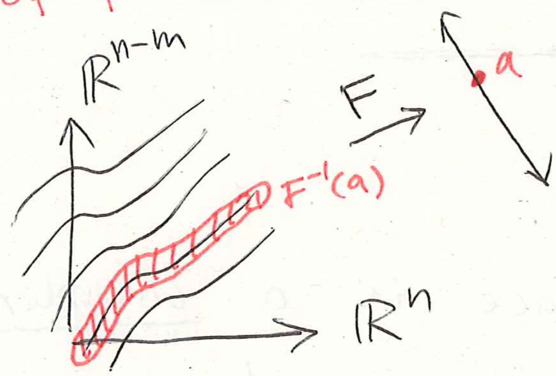
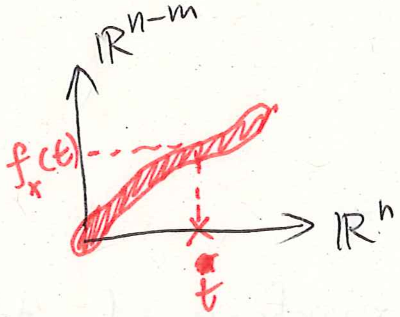
Such that

$$F^{-1}(a) \cap V_x = \Gamma_f$$ where Γ_f denotes the graph of f.

Think of $F^{-1}(a)$ as like contour lines



The theorem says locally, $F^{-1}(a)$ is like a graph from $\mathbb{R}^n$ to $\mathbb{R}^{n-m}$



A graph is locally isomorphic to the domain.

So eventually, $F^{-1}(a) \cap V_x \cong U_x \subset \mathbb{R}^n$

$F^{-1}(a)$ can be covered by open sets in $\mathbb{R}^n$. In fact, transition functions are smooth as well.

<u>Def</u>: Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a smooth fcn. A point $a \in \mathbb{R}^m$ is call a <u>regular value</u> for F, if for every $x \in \mathbb{R}^n$ s.t. $F(x) = a$, the differential $dF(x)$ is a surjective linear fcn.

<u>Thm</u>: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a sm. fcn. and $a \in \mathbb{R}^m$ a regular value for F. Then $F^{-1}(a)$ is a sm. manifold.

Riemann surfaces

<u>Def</u>: A Riemann surface is a complex analytic manifold of dimension 1.

(• locally $\mathbb{C}$
• transition functions are hol.)

Examples

• The ~~compact~~ Riemann surface of the square root

(Make the picture we drew more precise)

~~$V_1 = V_3 := \mathbb{C} \setminus \mathbb{R}^{\geq 0}$, $V_2 = V_4 := \mathbb{C} \setminus \mathbb{R}^{\leq 0}$~~

~~$\iota_i : V_i \to X$~~

~~$\iota_1(z_1) = z_1 \in X^1$~~

Skip this example?

Read the textbook

Graphs of complex function

$\boxed{\text{Read the text book}}$

Algebraic curves

By ~~Implicit~~ Holomorphic version of implicit function thm, if $f : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is a holomorphic function s.t. $0 \in \mathbb{C}^n$ is a regular value of f, then $f^{-1}(0)$ is a complex analytic mfd of dim 1.
R.S.

$f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^n$

$f = (f_1, \cdots, f_n)$ when each f_i is a polynomial, $f^{-1}(0)$ is also called affine algebraic curve.

When $n=1$, affine plane curve

Example: ($\mathbb{C}^2 = \{(x,y)\}$)
← coordinates

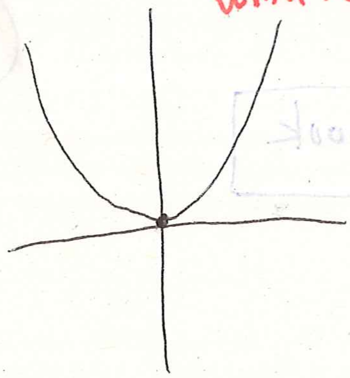
- Lines : $x=1$, $x+y=1$, $\cdots$

  $\{(x,y) \mid ax+by=c\} \cong \mathbb{C}$ if $(a,b) \neq (0,0)$

- Conic curves
  ← degree 2

  $\{(x,y) \mid y = x^2\}$

(caution: this is only a picture with real numbers!!)



$$\{(x,y) \mid y=x^2\} \longrightarrow \mathbb{C}$$
$$(x,y) \longmapsto x$$

Check this is an isomorphism as ~~holomorphic~~ complex analytic manifolds.

what about $X = \{(x,y) \mid x^2+y^2=1\}$ It's not a circle in $\mathbb{C}^2$!

Claim : $X \cong \mathbb{C} \setminus \{0\}$

$$x^2+y^2=1 \Rightarrow (x+\sqrt{-1}y)(x-\sqrt{-1}y)=1$$

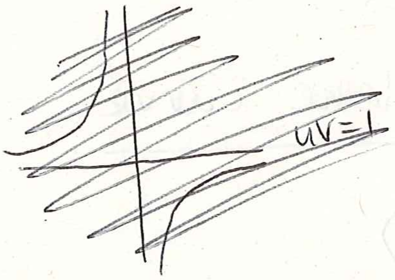
① Change of variable $u = x+\sqrt{-1}y$, $v = x-\sqrt{-1}y$

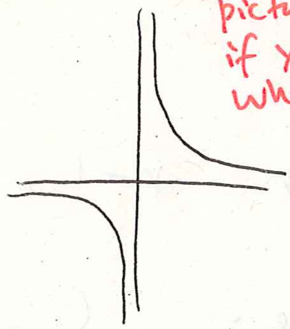
$$\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \quad \text{is an isomorphism}$$
$$(x,y) \longmapsto (u,v)$$

$$\phi\left(\{(x,y) \mid x^2+y^2=1\}\right) = \{(u,v) \mid uv=1\}$$

(Again, misleading picture. But useful if you understand what you're doing)

②





Check $\{(u,v) \mid uv=1\} \longrightarrow \mathbb{C} \setminus \{0\}$
$(u,v) \longmapsto u$

is an isomorphism.

(bijection
↓
homeomorphism
↓
analytical)