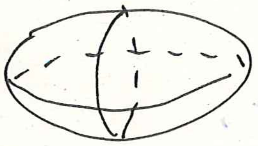
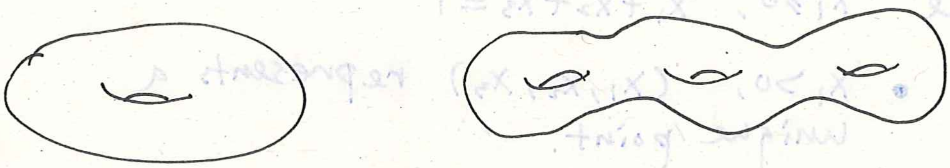
Compact Riemann surfaces

~~Fact~~: Any 2-dim compact orientable ~~$\blacksquare$~~ topological mfd can be given a complex structure (Not unique!)

 $= \mathbb{P}^1(\mathbb{C})$,

 ... are also complex manifolds. We will see who they are later.

differences between $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}^2(\mathbb{R})$ via sphere model:

$\mathbb{P}^1(\mathbb{C}) = \{(z_1, z_2) \in \mathbb{C}^2\}/\sim$

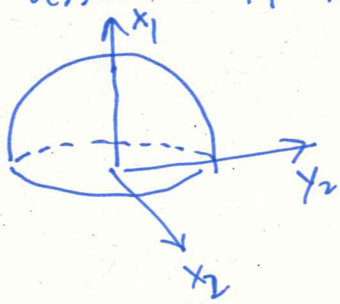
$z_1 = x_1 + \sqrt{-1}\, y_1$, $z_2 = x_2 + \sqrt{-1}\, y_2$, $x_i, y_i \in \mathbb{R}$

Given a $[z_1 : z_2]$, it is the same point as $[z_1\bar{z}_1 : z_2\bar{z}_1]$

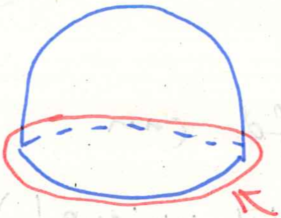
$= [\underbrace{|z_1|^2}_{\text{a real number!}} : z_2\bar{z}_1]$

Equivalence classes in $\mathbb{P}^1(\mathbb{C})$ can be represented by $(x_1, x_2 + \sqrt{-1}\, y_2)$ for some $x_1, x_2, y_2 \in \mathbb{R}$

By further scaling by real number, we may assume $x_1^2 + x_2^2 + y_2^2 = 1$ and $x_1 \geq 0$.



- When $x_1 > 0$, $(x_1, x_2 + \sqrt{-1}\, y_2)$ uniquely represents a point
- when $x_1 = 0$, any $(0, x_2 + \sqrt{-1}\, y_2)$ represents the same point due to complex number scaling.

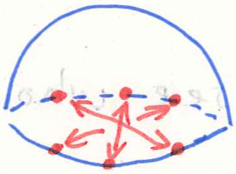


identify the boundary S^1 as a single point

Thus $\mathbb{P}^1(\mathbb{C}) = S^2$

For $\mathbb{P}^2(\mathbb{R})$,

Given (x_1, x_2, x_3), the same argument, we can assume $x_1 \geq 0$, $x_1^2 + x_2^2 + x_3^2 = 1$



- $x_1 > 0$, (x_1, x_2, x_3) represents a unique point.
- $x_1 = 0$, $(0, x_2, x_3) \sim (0, -x_2, -x_3)$

Caution: All arguments above need to ~~accompany with~~ be ~~slightly~~ modified in order to make a rigorous argument. Otherwise, it's just a proof of $\mathbb{P}^1(\mathbb{C}) \cong S^2$ as point sets. It's part of your topology training and it's your exercise!

Transition function of $\mathbb{P}^1(\mathbb{C})$:

Charts: $U_2 = \{[1 : x_2]\} \cong \mathbb{C}$

$U_1 = \{[x_1 : 1]\} \cong \mathbb{C}$

$U_1 \cap U_2 = \{[z_1 : z_2] \mid z_1 \neq 0, z_2 \neq 0\}$

under U_1: $x_1 = \frac{z_1}{z_2}$

under U_2: $x_2 = \frac{z_2}{z_1}$

$\boxed{x_1 = \frac{1}{x_2}}$

Transition fcn:

$$\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$$
$$x_1 \longmapsto x_2 = \frac{1}{x_1}$$

Complex Tori

Complex Tori are cpt. Riemann Surface of genus 1.

Defn: Let τ_1 and τ_2 be two cplx numbers ($\mathbb{R}$-linearly indept).

$$\Lambda = \{ n\tau_1 + m\tau_2 \mid n, m \in \mathbb{Z} \} \subset \mathbb{C}$$

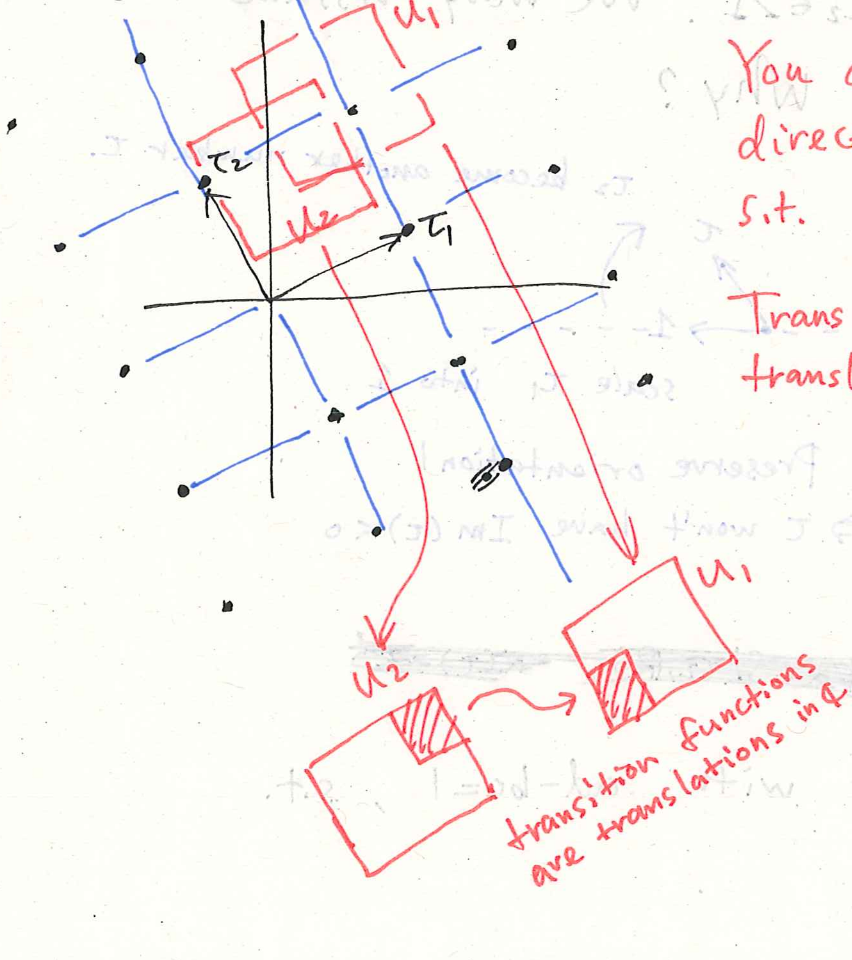
is called a lattice of cplx numbers.

Consider the quotient space $T = \mathbb{C}/\Lambda$ (in fact, quotient as groups.)

($z_1 \sim z_2$ iff $z_2 = z_1 + w$ for some $w \in \Lambda$)

$\mathbb{C} \xrightarrow{\pi} T$ projection, induce quotient topology

- It has a natural complex structure



You could define charts directly using open sets $U \subseteq \mathbb{C}$ s.t. $\pi(U) \cong U$ (no points are identified!)

Transition functions are merely translations in $\mathbb{C}$.

complex structure and lattice

Theorem: $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are biholomorphic (holomorphic str. is important here!)

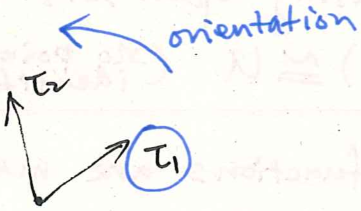
iff $\exists\, \alpha \in \mathbb{C}^*$ s.t. $\alpha \cdot \Lambda = \Lambda'$

Idea: $\Leftarrow$ obvious

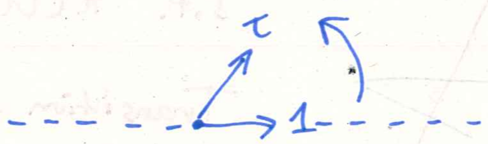
$\Rightarrow$
$$\begin{tikzcd} \mathbb{C} \arrow[r, dashed, "\tilde{f}"] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}/\Lambda \arrow[r, "f"] & \mathbb{C}/\Lambda' \end{tikzcd}$$

f lifts to $\tilde{f}$ b/c $\pi_1(\mathbb{C}) = 1$
holomorphic due to local ~~bih~~ homeo.
$\tilde{f}(\Lambda) \in \Lambda'$, $\tilde{f}$ bihol. $\Rightarrow \tilde{f}$ linear

~~We m~~
Pick two generators $\tau_1, \tau_2 \in \Lambda$. We may assume $\tau_1 = 1$ and $\mathrm{Im}(\tau_2) > 0$. Why?



τ_2 become another number τ.



scale τ_1 into 1

Preserve orientation!

$\Rightarrow \tau$ won't have $\mathrm{Im}(\tau) < 0$

Thm $\exists\, \alpha \in \mathbb{C}^*$, s.t.

$\alpha\{(1, \tau)\} \cong \{(1, \tau')\}$ [redacted]

iff $\exists\, a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, s.t.

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

$$\Rightarrow \quad \alpha^{-1}\tau' = a\tau + b$$
$$\alpha^{-1} = c\tau + d$$

$$\tau' = \frac{a\tau+b}{c\tau+d}$$

It is invertible, $\Rightarrow \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$

$(1, \tau) \rightsquigarrow (1, \tau')$ orientation preserving

$$\Rightarrow \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

$\Leftarrow$ Exercise

Fact: Any $\dim_{\mathbb{C}} 1$ cpt cplx mfd homeomorphic to a torus is bihol. to $\mathbb{C}/\Lambda$ for some lattice Λ.

$$\left\{ \begin{matrix} \text{genus } 1 \text{ cpt. Riemann} \\ \text{surfaces} \end{matrix} \right\} \xleftrightarrow{1:1} \mathbb{H}/SL(2, \mathbb{Z})$$

$$\mathbb{H} = \{ z \mid \mathrm{Im}(z) > 0 \}$$

$$SL(2, \mathbb{Z}) \curvearrowright \mathbb{H}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

Projective curves

Focus on $\mathbb{P}^2(\mathbb{C})$
$[z_0 : z_1 : z_2]$

Given a homogeneous polyn. $F(z_0, z_1, z_2)$
(say, $F = z_0^2 + z_1^2 + z_2^2$)

- Does F define a function $\mathbb{P}^2(\mathbb{C}) \to \mathbb{C}$?

No.

- Does $F(z_0, z_1, z_2) = 0$ ~~def~~ give us a well-defined ~~subs~~ closed subset in $\mathbb{P}^2(\mathbb{C})$?

Yes.

$$F(\lambda z_0, \lambda z_1, \lambda z_2) = \lambda^d F(z_0, z_1, z_2) \quad , \lambda \in \mathbb{C}^*$$
$(d = \deg F)$

λ^d does not affect the zero set.

~~Euler~~

(Want to use homogeneous polyn. to define a rich source of examples of compact cplx mfds)

<u>Euler identity</u>: F homog. polyn.

$$z_0 \frac{\partial F}{\partial z_0} + z_1 \frac{\partial F}{\partial z_1} + z_2 \frac{\partial F}{\partial z_2} = d F \quad , \quad (d = \deg F)$$

pf: Check each monomial.

Def: Given $F \in \mathbb{C}[z_0, z_1, z_2]$ a homog. polyn. of degree d.

$$V(F) := \{ [z_0 : z_1 : z_2] \in \mathbb{P}^2(\mathbb{C}) \mid F(z_0, z_1, z_2) = 0 \}$$

is called a plane projective curve of degree d.

If $\{ (z_0, z_1, z_2) \in \mathbb{C}^3 \mid \frac{\partial F}{\partial z_0} = \frac{\partial F}{\partial z_1} = \frac{\partial F}{\partial z_2} = 0 \} \subseteq \{ (0,0,0) \}$

then $V(F)$ is said to be smooth.

Prop: A smooth projective plane curve $V(F)$ is a cpt Riemann surface.

pf:
- check $V(F)$ is closed
- check each chart

$$U_1 = \{ (1, z_1, z_2) \subset \mathbb{C}^3 \}$$

$f_1(z_1, z_2) = F(1, z_1, z_2)$, $V(F) \cap U_1 = \{ f_1 = 0 \}$

$\frac{\partial}{\partial z_1} f_1 = \frac{\partial}{\partial z_1} F$, $\frac{\partial}{\partial z_2} f_1 = \frac{\partial}{\partial z_2} F$

$\boxed{\text{Argue by contradiction}}$

Both $= 0 \xRightarrow{\text{defn. of smooth}} \frac{\partial}{\partial z_0} F \Big|_{z_0 = 1} \neq 0$

Euler identity:

$$d \cdot F = \underbrace{z_0}_{=1} \frac{\partial F}{\partial z_0} + z_1 \underbrace{\frac{\partial F}{\partial z_1}}_{=0} + z_2 \underbrace{\frac{\partial F}{\partial z_2}}_{=0} \neq 0$$

, But $F(1, z_1, z_2) = 0$ contradiction! □

Example:

$$F(z_0, z_1, z_2) = z_0^2 + z_1^2 - z_2^2$$

$$\frac{\partial F}{\partial z_0} = 2z_0 \qquad \frac{\partial F}{\partial z_1} = 2z_1 \qquad \frac{\partial F}{\partial z_2} = -2z_2$$

$(0,0,0)$ the only solution $\Rightarrow$ smooth

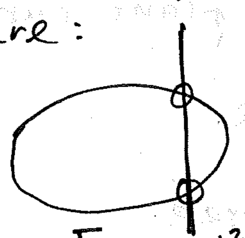
(side remark:

Take $U_3 = \{(z_0, z_1, 1)\}$,

$V(F) \cap U_3 = \{ z_0^2 + z_1^2 - 1 = 0 \}$

Recall $V(F) \cap U_3 \cong \mathbb{C} \setminus \{pt\}$

picture:



F $\quad$ $\mathbb{P}^2(\mathbb{C}) \setminus U_3$ infinity line

One may suspect, $V(F)$ is the one pt compactification of $\mathbb{C}$.)

Example (Elliptic curve)

$$F(z_0, z_1, z_2) = z_1^2 z_2 - (z_0 - \alpha_1 z_2)(z_0 - \alpha_2 z_2)(z_0 - \alpha_3 z_2)$$

$\alpha_1, \alpha_2, \alpha_3$ distict.

Smoothness:

$$\frac{\partial}{\partial z_1} F = 2 z_1 \cdot z_2 \quad , \quad \frac{\partial}{\partial z_1} F = 0 \Rightarrow z_1 = 0 \text{ or } z_2 = 0$$

$z_1 = 0$: $F = (z_0 - \alpha_1 z_2)(z_0 - \alpha_2 z_2)(z_0 - \alpha_3 z_2)$

solutions: $[z_0 : 0 : z_2] = [\alpha_1 : 0 : 1]$ or $[\alpha_2 : 0 : 1]$ or $[\alpha_3 : 0 : 1]$

α_i distinct $\Rightarrow \frac{\partial}{\partial z_0} F \neq 0$

$z_2 = 0$: $F = -z_0^3$,

solution: $[0 : 1 : 0]$

$$\left.\frac{\partial}{\partial z_2} F \right|_{z_0 = z_2 = 0} = 1 \neq 0$$