High dim generalization: projective variety
(read textbook)

Maps of Riemann surface

Holomorphic maps of R.S.

Def: Let X, Y be R.S. and $f: X \to Y$ a set function

1. f is holomorphic at $x \in X$ if for every choice of charts $\varphi_x, \varphi_{f(x)}$ the function $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ is holomorphic at x

2. If $U \subset X$ is open, f is hol. on U if f is hol. at each $x \in U$

3. ~~If~~ f is hol. on $U = X$, we say f is a holomorphic map

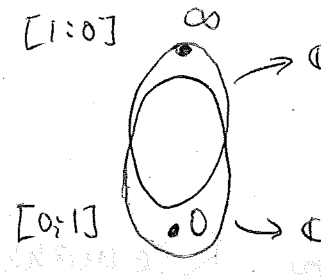
$$\begin{tikzcd} U_x \arrow[r, "f"] & U_{f(x)} \arrow[d, "\varphi_{f(x)}"] \\ V_x \arrow[u, "\varphi_x^{-1}"] \arrow[r, "\varphi_{f(x)} \circ f \circ \varphi_x^{-1}"'] & V_{f(x)} \end{tikzcd}$$

$\underline{\varphi_{f(x)} \circ f \circ \varphi_x^{-1}}$ local expression

Ex

$\mathbb{P}^1 \longrightarrow \mathbb{P}^1$

$(\mathbb{P}^1(\mathbb{C}) = \mathbb{P}^1$ abbreviation$)$



$$U_1 = \{[x_1 : 1]\} \xrightarrow[f(x_1) = x_1^2]{\cancel{f(z)=z^2}} U_1$$

• Does f extend to ∞ as a hol. fcn?

~~Under $U_2 = \{[1 : x_2]\}$,~~
~~$f(x_2) = \frac{1}{x_2}$~~

$$\begin{tikzcd}
 & U_1 \arrow[rr, "x_1 \mapsto x_1^2"] & & U_1 & \\
U_1 \cap U_2 \arrow[ur] \arrow[dr] \arrow[rrrr] & & & & U_1 \cap U_2 \arrow[ul] \arrow[dl] \\
 & U_2 \arrow[rr, "?"'] & & U_2 &
\end{tikzcd}$$

$$\boxed{\begin{tikzcd}
x_1 \arrow[d, mapsto] & U_1 \cap U_2 \arrow[r, "x_1 \mapsto x_1^2"] \arrow[d] & U_1 \cap U_2 \arrow[d] & x_1 \arrow[d, mapsto] \\
x_2 = \frac{1}{x_1} & U_1 \cap U_2 \arrow[r] & U_1 \cap U_2 & \frac{1}{x_1} = x_2 \\
 & x_2 = \frac{1}{x_1} \arrow[r, mapsto] & \frac{1}{x_1^2} = x_2^2 &
\end{tikzcd}}$$

Yes.

Exercise: $f(z) = (z-a)(z-b)(z-c)$ for $a, b, c \in \mathbb{C}$

Prove f extends to ∞ as hol. fcn.

~~Def~~

Isomorphism: invertible, hol. inverse is also hol.

Automorphism: $f: X \longrightarrow X$ where f is an isomorphism.

Local structure of Maps

a chart (U_x, φ_x) for a R.S. X is centered at x, if $\varphi_x(x) = 0$

Thm: Let $f: X \longrightarrow Y$ be a non const. hol. map of R.S.. For any $x \in X$, there are charts centered at x, $f(x)$, such that the local expression of f using these chart is $z \mapsto z^k$ for some $k \in \mathbb{N}_{\geq 1}$

Pf: Pick any ~~chat~~ charts φ, ψ centered at x, $f(x)$

$F = \psi \circ f \circ \varphi^{-1}$ local expression

$F = z^k \left(\sum_{n=0}^{\infty} a_{k+n} z^n \right)$ Taylor expansion, $a_k \neq 0$.

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U \arrow[u, phantom, "\cup"] \arrow[d, "\varphi"] & U' \arrow[u, phantom, "\cup"] \arrow[d, "\psi"] \\
V \arrow[r, "F=h^k"] \arrow[d, "h"] & V' \\
\tilde{V} \arrow[ur, dashed, "\tilde{z}^k = h^k(z)"'] &
\end{tikzcd}$$

$\tilde{z} = h(z)$

Let $G = \sum_{n=0}^{\infty} a_{k+n} z^n$,

G is holomorphic at 0, $G(0) \neq 0$

$\Rightarrow \sqrt[k]{G}$ is defined around 0

$h := z \sqrt[k]{G}$, $h^k = F$, $h'(0) \neq 0$

$\Rightarrow h$ isomorphism near 0

$\square$

Exercise: the integer k is unique.

Def: Let $f: X \to Y$ be a non-const. hol. map of R.S.

- Given $x \in X$, the integer k_x s.t. $\exists$ local expr. $F(z) = z^{k_x}$, is called the ramification index of f at x.
- The quantity $\nu_x = k_x - 1$ is called the differential length of f at x
- If a point x has ramification index $k_x = 1$, $\Rightarrow$ f is unramified at x
- A point x s.t. $k_x \geq 2$ is called ramification point. The ramification locus R is the subset of X consisting of all ramification points
- If x is a rami. pt., then $f(x) \in Y$ is called a branch pt. The branch locus $B = \{\text{branch pt} \in Y\}$

Rmk: - rami. locus $\neq$ inverse image of branch locus

- f is unramified at $x \in X$ iff $\forall$ local expr F of f around x, $F'(\varphi_x(x)) \neq 0$

- $f: X \to Y$, ramified at $x \in X$ ($k_x > 1$)
  $\exists$ nbh. $U \subset X$ s.t. $\forall x' \in U$, $k_{x'} = 1$ whenever $x' \neq x$

Why?

Check local expr. $\underline{x^{k_x}}$

Lem: $R \subset X$ is discrete.

Cor: If X is cpt, $\Rightarrow$ R is finite

(~~Interesting exercise: Prove $\forall \mathbb{P}^1 \to \mathbb{C}$ hol map must be constant.~~)

Maps of compact R.S.

Thm: Let $f: X \to Y$ be a holomorphic map of R.S, X cpt. If f is non-constant, then f is onto.

pf. If f is nonconst, $f(X)$ is open (open map)
X is cpt $\Rightarrow$ $f(X)$ is closed
(Riemann Surfaces are by defn connected)
Thus $f(X) = Y$

$\square$

~~Interesting exercise.~~

<u>Cor</u>: X cpt R.S.

$f: X \longrightarrow \mathbb{C}$ hol. $\Rightarrow$ f is constant

Fibers of hol. maps

$f: X \longrightarrow \underset{\substack{\uplus \\ y}}{Y}$, X, Y cpt, f hol.

Let $y \in Y$, $x \in f^{-1}(y)$

Because of local expr. $F = x^{k_x}$,

$f^{-1}(y)$ is discrete, thus <u>finite</u>

<u>Thm</u>: Let $f: X \to Y$ be non const hol. map of cpt. R.S. If $y_0, y_1 \in Y$ are not in branch locus B, then $|f^{-1}(y_0)| = |f^{-1}(y_1)|$

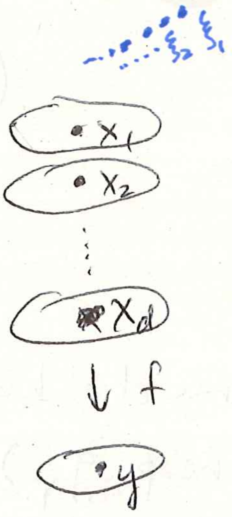
pf. $d = |f^{-1}(y_0)|$

<u>Claim</u> $A = \{ y \in Y \setminus B \mid |f^{-1}(y)| = d \}$ is open

$y \in A$, $f^{-1}(y) = \{ x_1, \cdots, x_d \}$

near each x_i, there is a local expr. $F(x_i) = x_i$

local isom.



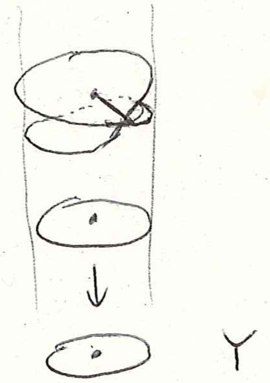
If there were other points $\{\xi_n\}$ ~~the preimage of~~ s.t. $\{f(\xi_n)\} \ni y$, by compactness of X, there is a limit s.t. $f(\xi)=y$, contradiction

Apply the same argument to $Y \setminus A$. $\square$

Def: For $f: X \to Y$ a non-const. hol. map of cpt R.S., the degree of f is the cardinality of the fiber of any point $y \in Y \setminus B$. If f is const., we say f has deg 0.

Exercise: Let $f: X \to Y$ hol. of cpt R.S. degree $d > 0$. $y \in B \subset Y$ and $f^{-1}(y) = \{x_1, \cdots, x_n\}$. Show

$$\sum_{i=1}^{n} k_{x_i} = d$$



- Use local expression!
- Argue there ~~can't be~~ exists $U \subset Y$ s.t. $f^{-1}(U)$'s ~~has~~ components ~~that~~ are nbhds of $\{x_1, \cdots, x_n\}$
- count.

Fill out details by yourself.

Riemann-Hurwitz formula

Thm: (R-H formula)

Let $f: X \to Y$ be a non const. degree d, hol. map of cpt R.S. Denote by g_X (resp. g_Y) the genus of X (resp Y). Then

$$2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X}' v_x$$

finite sum because $v_x \neq 0$ only if $x \in R$

where $v_x = k_x - 1$ is the differential length of f at x.