Pf of R-H Formula:

Euler char of a cpt orientable surface:

$$\chi(X) = 2 - 2g(X)$$

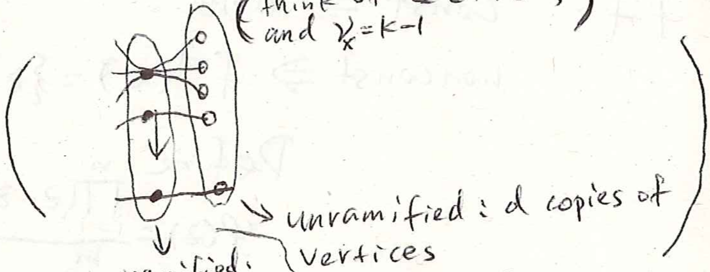
On the other hand, $\chi(X) = V - E + F$ for a good graph on X

Fix a good (suitable) graph on Y, lift it to a good graph on X. (taking preimage)

"Suitable": Vertex contains branch locus B, ~~edges don't~~

~~#V(Y)~~

- $V(X) = dV(Y) - \sum_x \nu_x$



- $E(X) = d\,E(Y)$

- $F(X) = dF(Y) \leftarrow$

(a face $\cong$ unit disc. $\pi_1 = 1$, $\Rightarrow$ coverings are trivial)

$$\chi(X) = d\chi(Y) - \sum_x \nu_x$$

(the ~~network of~~ lifted graph is in general complicated. But we don't care.)

$$\Rightarrow 2 - 2g(X) = d(2 - 2g(Y)) - \sum_x \nu_x \quad \square$$

Ex: $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$, f the extension of $z \mapsto z^k$

Ramified pts: $\{0, \infty\}$

Diff. lengths : $k-1,\ k-1$

Degree : k.

$$2 = 2k - (k-1) - (k-1) \checkmark$$

Example of maps between cpt R.S.

- Given a rational fcn $f = \frac{p(z)}{q(z)}$, if $p(z), q(z)$ do not have common zero, f extends to a map $\mathbb{P}^1 \to \mathbb{P}^1$. We use rat'l fcn to directly refer to such extension.

Thm If $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a hol. map of R.S. Then f is a rat'l fcn : $f = \frac{p(z)}{q(z)}$, $p(z), q(z) \in \mathbb{C}[z]$

Pf: Const $\Rightarrow$ done.

nonconst $\Rightarrow f^{-1}(0) = \{z_1, \cdots, z_n\}$, $f^{-1}(\infty) = \{p_1, \cdots, p_m\}$.

Define
$$\varphi(z) = \frac{\prod_{i=1}^{n} (z - z_i)^{k_{z_i}}}{\prod_{j=1}^{m} (z - p_j)^{k_{p_j}}}$$
where k_{z_i}, k_{p_j} are rami indices

(assume no z_i, p_j is ∞ by linear transformation)

Check: $f(z)/\varphi(z)$ does not take value 0 or ∞

The only possible zero is when $z = \infty$
— — — — — — — ∞ is when $z = \infty$
but impossible having both

nonsurjective hol. $\Rightarrow$ const.

Maps of elliptic curves

$P(X,Y,Z) = Y^2Z - (X-a_1Z)(X-a_2Z)(X-a_3Z)$ $\quad (\mathbb{P}^2 = \{[X:Y:Z]\})$

- $V(P)$: Elliptic curve
- when a_1, a_2, a_3 are distinct, $V(P)$ is smooth

Lem: $g(V(P)) = 1$

pf: consider the chart $(\frac{X}{Z}, \frac{Y}{Z}) = U_1$ $\quad (u = \frac{X}{Z},\ v = \frac{Y}{Z})$

$$P = 0 \iff \left(\frac{Y}{Z}\right)^2 - \left(\frac{X}{Z} - a_1\right)\left(\frac{X}{Z} - a_2\right)\left(\frac{X}{Z} - a_3\right) = 0$$

$\pi : U_1 \cap V(P) \longrightarrow \mathbb{C} \subset \mathbb{P}^1$ is a degree 2 map
$\quad (u,v) \longmapsto [u:1]$

On another chart $(\frac{X}{Y}, \frac{Z}{Y}) = U_2$ $\quad (u' = \frac{X}{Y},\ v' = \frac{Z}{Y})$

$$P = 0 \iff v' - (u'-a_1v')(u'-a_2v')(u'-a_3v') = 0$$

Heuristically
$\left(\frac{1}{v'^2} = (\frac{u'}{v'} - a_1)(\frac{u'}{v'} - a_2)(\frac{u'}{v'} - a_3) \right.$
$\left. \text{so } v' \to 0,\ \frac{u'}{v'} \to \infty \right)$

π ~~were able to extend to U_2~~ on U_2 becomes

$U_2 \cap V(P) \longrightarrow \mathbb{P}^1$
$(u', v') \longmapsto [\frac{u'}{v'} : 1]$ when $(u',v') \neq 0$

Fact:
- The map can be extended to $u' = v' = 0$, and $(0,0) \longmapsto [1:0]$

(or use Riemann existence thm in Thm 6.2.2)

~~What are the rami. pts?~~

Check branch locus $B = \{a_1, a_2, a_3, \infty\}$, $\#R = 4$ b/c degree 2

(check Jacobian of the map)

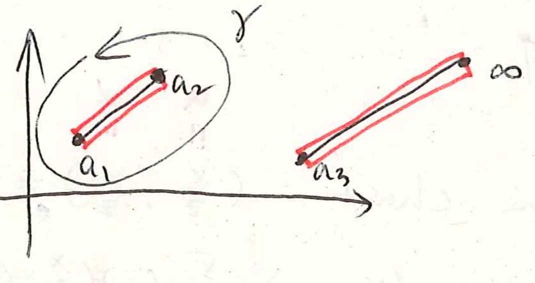
$$2 - 2g_{V(P)} = 2 - 0 - \sum_{p \in R} 1 = -2$$

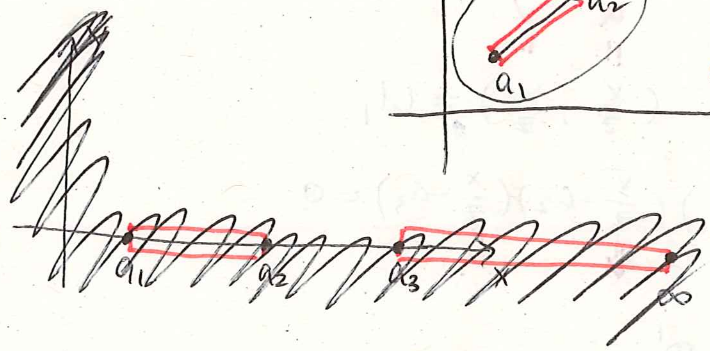
$$g_{V(P)} = 1$$

□

Extra: Another way to see elliptic curve as a torus.

$$\{ y^2 - (x-a_1)(x-a_2)(x-a_3) \} \subset \mathbb{C}^2$$
$$\downarrow$$
$$\{(x)\} = \mathbb{C}$$





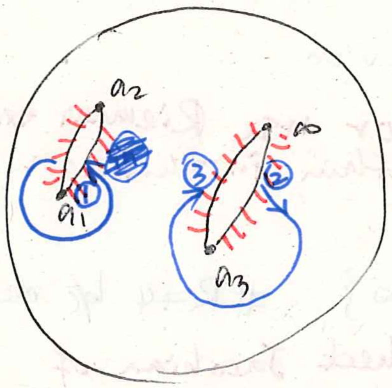
Take out line segments between pairs of two pts (make sure there is no intersection)

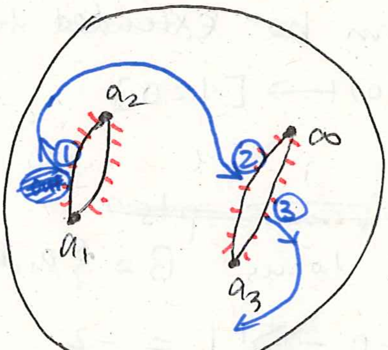
$$\int_\gamma \frac{dx}{(x-a_1)(x-a_2)(x-a_3)} = 0 \Rightarrow$$ the loop $p(\gamma)$ does not contain 0

where $p(x) = (x-a_1)(x-a_2)(x-a_3)$

So $\sqrt{p(x)}$ can be defined away from the two segments.

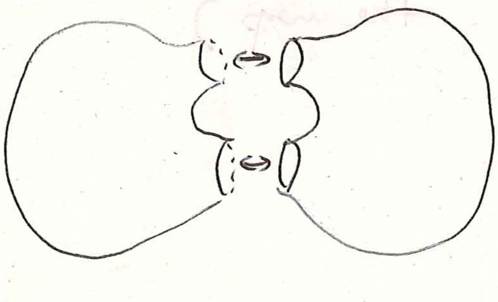
(need more rigorous arguments)





glue two pieces of spheres together

A path could looks like what is drawn in blue.



Loops and lifts

Maps $f, g : X \to Y$

- A homotopy between f and g:

$$H : X \times [0,1] \to Y \quad (\leftarrow \text{time})$$

s.t. $H(x,0) = f(x)$, $H(x,1) = g(x)$ $\forall x \in X$

- If a htpy exists, say f, g are homotopic ($f \sim g$)
- Let $A \subseteq X$ be such that $f|_A = g|_A$. A htpy H between f and g is said to be relative to A, if $\forall a \in A$ $\forall t \in [0,1]$, $H(a,t) = f(a) = g(a)$

<u>Defn</u> Two topological spaces X, Y are called homotopy equiv. (or simply homotopic) and denoted by $X \sim Y$, if $\exists f : X \to Y$, and $g : Y \to X$ s.t.

$g \circ f \sim Id_X$, $f \circ g \sim Id_Y$

~~Fundamental group~~

<u>Ex</u>: • $\mathbb{R} \rightleftarrows \{pt\}$

$0 \longleftarrow\!\!| \ pt$

$$H : \mathbb{R} \times [0,1] \to \mathbb{R}$$
$$(x, t) \mapsto (xt)$$

$H(x,0) = 0$, $H(x,1) = x$

(a top sp. htpy equiv to $\{pt\}$: <u>contractible</u>)

The fundamental gp

Def: Let X be a top sp and $x_0 \in X$. A loop in X with base pt x_0 is a continuous map $\gamma : [0,1] \to X$ s.t. $\gamma(0) = \gamma(1) = x_0$

Two loops γ, δ with base pt x_0 are said to be homotopic wrt base pt if $\exists$ htpy $H : [0,1] \times [0,1] \to X$ between γ and δ s.t. $\forall t \in [0,1]$, $H(0,t) = H(1,t) = x_0$ (denoted by $\gamma \sim \delta$)

Ex: loops being homotopic wrt base pt is an equivalence relation

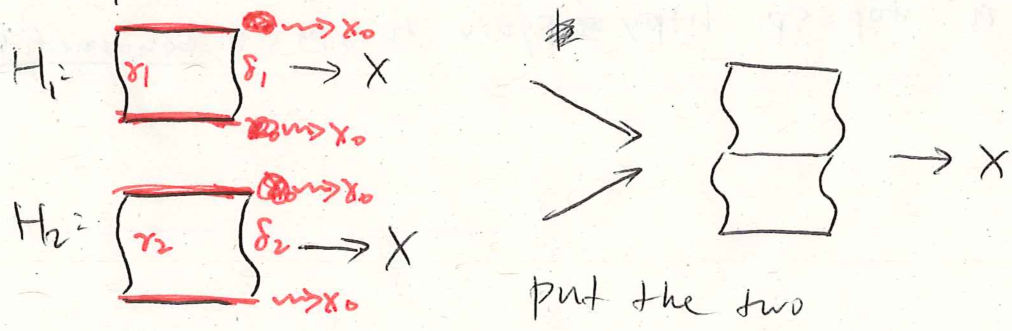
Def: Given two loops γ_1, γ_2 with base pt x_0, define $\gamma_1 * \gamma_2$ in X w/ base pt x_0 as

$$\gamma_1 * \gamma_2 (s) = \begin{cases} \gamma_1(2s) & \text{if } s \in [0, \frac{1}{2}] \\ \gamma_2(2s-1) & \text{if } s \in [\frac{1}{2}, 1] \end{cases}$$

Lem: "$*$" and homotopy equiv. commutes. i.e. if $\gamma_1 \sim \delta_1$ and $\gamma_2 \sim \delta_2$

$$\gamma_1 * \gamma_2 \sim \delta_1 * \delta_2$$

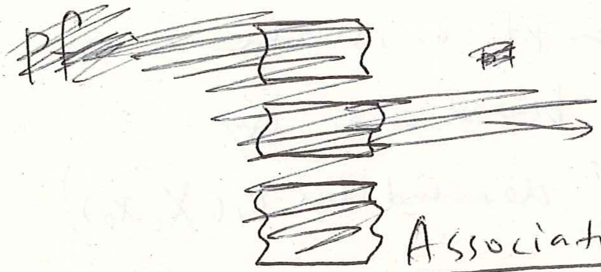
pf.



put the two homotopy together.

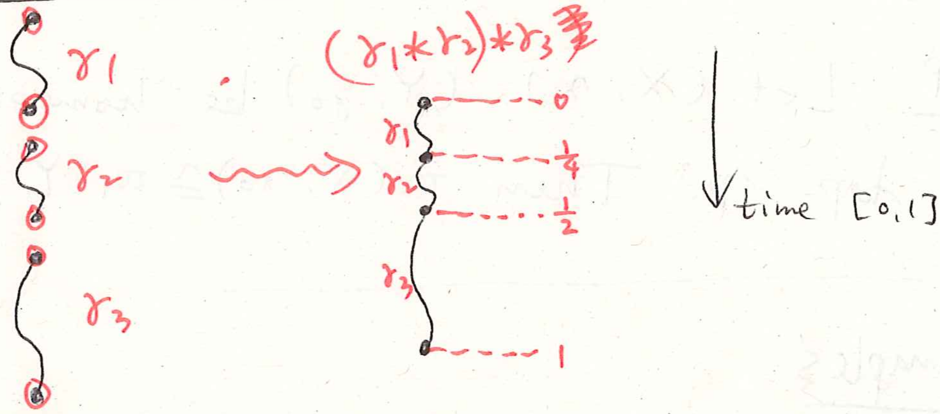
Ex: fill in details □

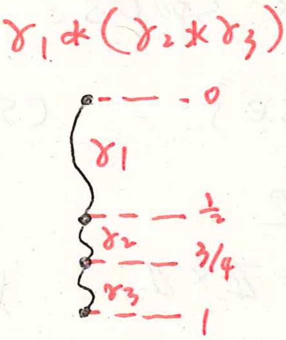
Thm: Let X be a topol. sp and $x_0 \in X$.
Then the set of equiv. classes of loops w/ base pt. x_0 is a group under "$*$"

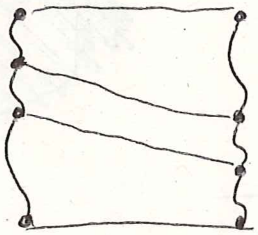


Sketch of pf:

Associativity:







homotopy between $(\gamma_1 * \gamma_2) * \gamma_3$ and $\gamma_1 * (\gamma_2 * \gamma_3)$ is scaling the time.

Identity: constant loop.

(Check by yourself)

Inverse: $\gamma^{-1}(s) := \gamma(1-s)$. $s \in [0,1]$
(Check it is an inverse)

Def: Let X be a topol. sp. and $x_0 \in X$. The fundamental gp. of X w/ base pt x_0 is the group of equiv. classes of loops based on x_0, w/ operation induced by "$*$" (denoted by $\pi_1(X, x_0)$)

Prop. Let (X, x_0), (Y, y_0) be homeomorphic pointed top. sp. Then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$

Examples

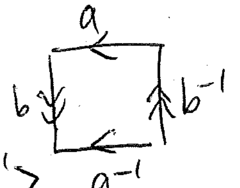
Contractible spaces: $\{\cdot\}$ $\mathbb{R}^n$

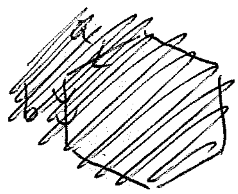
$\pi_1(S^2) = \{e\}$ (simply connected)

$\pi_1(S^1) = \mathbb{Z}$

$\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$

$= \langle a, b \mid aba^{-1}b^{-1} \rangle$





~~Check~~

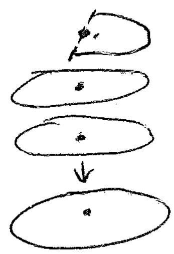
Ex: Given a cont. map $f: X \to Y$, $x_0 \in X$, $y_0 \in Y$ s.t. $y_0 = f(x_0)$

f induces a gp homomorphism $\pi_1(f): \pi_1(X, x_0) \to \pi_1(Y, y_0)$

Covering spaces

Def: A covering is a continuous, surjective map $p: X \to Y$, s.t. $\forall\, y \in Y$ and each $x_i \in p^{-1}(y)$, $\exists$ a nbh. U_y of y, s.t. $p^{-1}(U_y) =$ disjoint ~~union~~ nbh. V_{x_i}, and $p|_{V_{x_i}}: V_{x_i} \to U_y$ is a homeomorphism

e.g.:



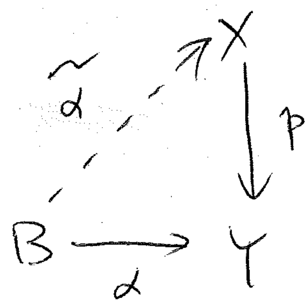
(Side note:
local homeo : "local on the domain"
covering : "local on the target")

Example: $f: X \to Y$ hol. map of cpt R.S.

$R \subset X$ ramification locus, $B \subset Y$ branch locus

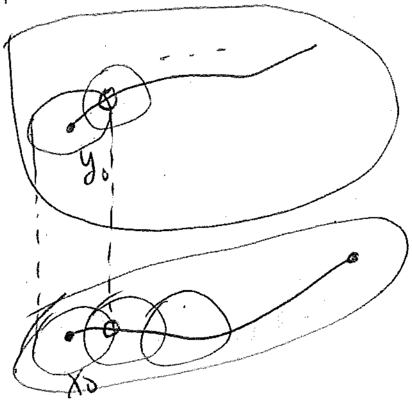
$f|_{X \setminus R} : X \setminus R \to Y \setminus B$ is a covering map

Def: Given a covering $p: X \to Y$, and a cont. fcn: $\alpha: B \to Y$, a lift of α is a cont. fcn $\tilde{\alpha}: B \to X$ s.t. $p \circ \tilde{\alpha} = \alpha$.



Lem (Path lifting) Let $p: X \to Y$ be a covering. If $\alpha: [0,1] \to Y$ is a path s.t. $\alpha(0) = y_0$ and $x_0 \in p^{-1}(y_0) \subset X$, then $\exists!$ lift $\tilde{\alpha}: [0,1] \to X$ s.t. $\tilde{\alpha}(0) = x_0$

Sketch of pf.



use finitely many open sets to cover the path, and lift the path segment by segment.

<u>Lem</u>: Let $p: X \to Y$ be a covering and $H: [0,1] \times [0,1] \longrightarrow Y$ a homotopy between paths $\alpha, \beta: [0,1] \to Y$ relative to endpoints.
Let $y_0 = \alpha(0) = H(0,0)$, $x_0 \in p^{-1}(y_0) \subset X$
Then $\exists$ a lifting $\tilde{H}: [0,1] \times [0,1] \longrightarrow X$ of H s.t. $\tilde{H}(0,0) = x_0$

(similar strategy)

<u>Cor</u>: $p: X \to Y$ covering map.
~~$\pi_1(X)$~~
the induced $\pi_1(p)$ is injective.

Def ~~(U, u_0)~~

Given a covering $q : (U, u_0) \to (Y, y_0)$, if U is simply connected ($\pi_1(U) = \{e\}$), then q is called universal cover of (Y, y_0)

Facts:

- Universal cover always exists
- – – – – – – is unique up to homeomorphism.

Example:

- $\mathbb{R} \longrightarrow S^1$ is a univ. covering
  $\theta \longmapsto e^{i\theta}$
- $\mathbb{C} \longrightarrow T \cong \mathbb{C}/\Lambda$ is a univ. covering

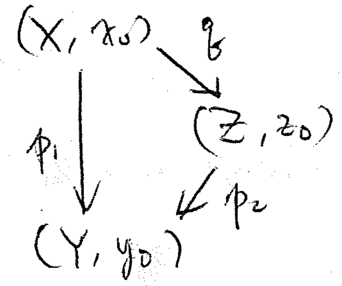
Galois correspondence:

Fix (Y_0, y_0)

$$\left\{ \begin{array}{l} \text{path connected} \\ \text{covers } (X, x_0) \to (Y, y_0) \end{array} \right\} \overset{\cong}{\longleftrightarrow} \{ \text{subgps of } \pi_1(Y, y_0) \}$$

$$[p : X \to Y] \longmapsto \pi_1(p)\big(\pi_1(X, x_0)\big) < \pi_1(Y, y_0)$$

poset structure is preserved:



$$\Longleftrightarrow \quad \pi_1(p_1)\big(\pi_1(X, x_0)\big) < \pi_1(p_2)\big(\pi_1(Z, z_0)\big) < \pi_1(Y, y_0)$$

(also π_1 is isomorphic to group of deck transf....)

§ Counting maps

Defn: Two hol. maps of R.S. $f: X \to Y$ and $g: \tilde{X} \to Y$ are called *isomorphic* if $\exists$ isomorphism of R.S. (bihol.) $\phi: X \to \tilde{X}$ s.t. $f = g \circ \phi$

An *automorphism* of $f: X \to Y$ is an iso. $\psi: X \to X$ s.t. $f = f \circ \psi$.

The group of automorphism : $\mathrm{Aut}(f)$.

Recall the first class: we count maps weighted by # of automorphisms. And we ran into the confusion about the fact $\mathrm{Aut}(f) = \{1\}$ where $f: S_1 \amalg S_1 \to S_1$, with $z \mapsto z$ on one copy and $z \mapsto z^2$ on the other.

Example: Affine elliptic curve $E = V(y^2 - (x-a_1)(x-a_2)(x-a_3))$

$$\pi: E \longrightarrow \mathbb{C}$$
$$(x,y) \longmapsto x$$

The map $E \to E$, $(x,y) \mapsto (x,-y)$ is a nontrivial automorphism of π.

(swap fibers!)

Defn $d \in \mathbb{Z}_{>0}$. A partition of d is an unordered tuple of positive integers $\lambda = (k_1, k_2, \cdots)$ s.t. $\sum k_i = d$ (some k_i may be the same)

The size of λ: d.

The length of λ: # of elements in λ.

An automorphism of λ: ~~$\phi: \{1, \dots, k\} \to \{1, \dots, k\}$~~ ~~bijection $\phi: \lambda \to \lambda$ s.t. $\forall i$.~~ ~~the equality of integer $\phi(k_i) = k_i$~~ ~~holds~~ ~~bijection ϕ~~

Fix an order of elements in λ as $(k_1, k_2, \cdots)$.
It is a bijection $\phi: \lambda \to \lambda$ s.t. $\forall i$. $\phi(k_i) = k_i$ as integers.

Example: $d = 3$.

(3), $(2, 1)$, $(1, 1, 1)$

Aut: $\{1\}$, $\{1\}$, S_3

(like
(1, 1, 1)
↓ ↓ ↓
(1, 1, 1)).

Defn. Let $f: X \to Y$ be a hol. map of R.S. of degree d. Let $y \in Y$ and let $f^{-1}(y) = \{x_1, \dots, x_n\}$. ~~Recall that~~

Let k_{x_i} be the ramification index of f at x_i (locally $z \mapsto z^{k_{x_i}}$)

Call the ~~set~~ partition $\{k_{x_1}, \dots, k_{x_n}\}$ the ramification profile of f at y.

f unramified over y : $(k_{x_1}, \dots, k_{x_n}) = (1, 1, \dots, 1)$

f has simple ramification : $(k_{x_1}, \dots, k_{x_n}) = (2)$ or $(2, 1, 1, \dots, 1)$

f is fully ramified : $(k_{x_1}, \dots, k_{x_n}) = (d)$ where d is the degree of f.

Example: $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$

$[z_0 : z_1] \longmapsto [z_0^d : z_1^d]$

fully ramified over 0 and ∞ (or $[0:1]$ and $[1:0]$)