the sum runs over each isom class of $f: X \to Y$ s.t.

1) f is hol. map between R.S.
2) X connected, cpt, and has genus h.
3) the branch locus is $\{b_1, \cdots, b_n\}$
4) the ramification profile of f at b_i is λ_i

We call a map f satisfying 1–4 a <u>Hurwitz cover</u> for the discrete data $g, h, d, \lambda_1, \cdots, \lambda_n$

<u>Rmk</u>: For Hurwitz cover to exist, R-H formula must hold

(note also $\sum_{x \in X} \nu_x = nd - \sum_{i=1}^{n} \ell(\lambda_i)$, so R-H $\Rightarrow 2h-2 = d(2g-2) + nd - \sum_{i=1}^{n} \ell(\lambda_i)$)

<u>Ex</u>: Let $Y = \mathbb{P}^1$ and set $b_1 = 0$, $b_2 = \infty$. Choose $d > 0$ and let $\lambda_1 = \lambda_2 = (d)$.

$$H^{d}_{0 \to 0}((d),(d)) = \frac{1}{d}$$

pf:
$$p: \mathbb{P}^1 \to \mathbb{P}^1$$
$$[x:y] \mapsto [x^d, y^d]$$

<u>Claim</u>: any Hurwitz cover $f: X \to \mathbb{P}^1$ is isomorphic to p.

- $g(X) = 0$, X sm. cpt $\Rightarrow X \cong \mathbb{P}^1$
  so $f: \mathbb{P}^1 \to \mathbb{P}^1$, thus f is a rational function

- $f(r_1)=0$, $f(r_2)=\infty$ where r_1, r_2 are ramified pts.

Assume neither r_1, r_2 is ∞ (use a linear trans.)

then

$$f(x) = b\frac{(x-r_1)^d}{(x-r_2)^d}, \quad b\in\mathbb{C} \qquad \left(f(x)\cdot\frac{(x-r_2)^d}{(x-r_1)^d} \text{ has no zero no pole} \right)$$

$$= \left(a\frac{(x-r_1)}{(x-r_2)} \right)^d \quad \text{for some } a\in\mathbb{C}$$

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, "\phi"] \arrow[rr, bend right, "f"'] & \mathbb{P}^1 \arrow[r, "p"] & \mathbb{P}^1 \\
{[x:1]} & {[y:1]} & {[z:1]}
\end{tikzcd}$$

$a\frac{x-r_1}{x-r_2}=y$ $\qquad z=y^d$

$f = p\circ\phi$, ϕ is an isomorphism of $\mathbb{P}^1$,

thus $f\sim p$

Claim $|\mathrm{Aut}(p)| = d$

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[dr, "x^d"] & \\
& \mathbb{P}^1 \\
\mathbb{P}^1 \arrow[uu, "\phi"] \arrow[ur, "x^d"'] &
\end{tikzcd}$$

$\phi(0)=0$, $\phi(\infty)=\infty$

on the affine chart, there is no pole, and 0 is the only zero.

$\phi(x) = ax^k$, $a\in\mathbb{C}$

$(ax^k)^d = x^d \Rightarrow a^d=1$ and $k=1$

$a\in\mu_d$, group of d-th roots of unity

$$\mathrm{Aut}(p) = \mathbb{Z}/d\mathbb{Z}$$

Def: $H^{\bullet}_{h \xrightarrow{d} g}(\lambda_1, \dots, \lambda_n)$ ~~counts~~

$$= \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|}$$ but counts possibly disconn. domain

Note: definition of genus of disconn. curve is different:

$$\chi(X \sqcup Y) = \chi(X) + \chi(Y)$$

Want to keep $\chi(X \sqcup Y) = 2 - 2g(X \sqcup Y)$

thus we need to define:

$$g(X \sqcup Y) = g(X) + g(Y) - 1$$

Ex: $g(\mathbb{P}^1 \sqcup \mathbb{P}^1) = 0 + 0 - 1 = -1$! (disconnected genus can be negative!)

~~HW~~

Exercise: using disconnected genus, R-H formula remains unchanged.

Riemann Existence thm

$$\{\text{Topological cover of punctured R.S.}\} \Leftrightarrow \{\text{hol. ramified cover of R.S.}\}$$

Def: A continuous function between cpt top. surfaces $p: X \to Y$ is called a ramified cover if there is a finite set $B \subset Y$ s.t.

- $p^{-1}(B) \subset X$ is finite
- $p: X \setminus p^{-1}(B) \to Y \setminus B$ is a covering.

Thm (Riemann's existence thm)

Let Y be a cpt R.S. and X° a topological surface. Assume that there are a finite number of pts $b_1, \cdots, b_n \in Y$ and a fcn. $f^\circ : X^\circ \to Y \setminus \{b_1, \cdots, b_n\}$ which is a top. cover of finite degree. Then there exist a unique (up to isom) cpt. R.S. X which contain X° as a dense open set s.t. f° extends to $f : X \to Y$, a hol. map of R.S.

~~Sketch of pf:~~

(Sketch of pf see text book)

Hyperelliptic covers

Def: A Riemann surface X is called hyperelliptic if it admits a hol. map f to $\mathbb{P}^1$ of degree 2. f is called a hyperelliptic cover

$$f : X \to \mathbb{P}^1$$

$$2g(X) - 2 = 2(-2) + \sum_{x \in X}' \nu_x \Rightarrow 2g(X) + 2 = \sum_{x \in X}' \nu_x$$

If $x \in X$ is a rami. pt. $\Rightarrow \nu_x = 1$ (0, 1 are the only choices)

so there are $2g(X)+2$ rami. pts.

- $H_{g \geq 0}((2)^{2g+2})$

$$\underbrace{(2), (2), \cdots, (2)}_{2g+2}$$

$X \longrightarrow \mathbb{P}^1$

Fix $b_1, \dots, b_{2g+2} \in \mathbb{P}^1$ distinct.
We may assume they are different from ∞

Consider $X^\circ \subset \mathbb{C}^2$ defined by

$$y^2 = \prod_{i=1}^{2g+2} (x - b_i)$$

$$p^\circ : X^\circ \longrightarrow \mathbb{C}$$
$$(x, y) \longmapsto x$$

Using Riemann existence thm, p° extends uniquely to a map $p : X \longrightarrow \mathbb{P}^1$

$\deg(p) = 2$, since $\sum V_x$ is even, ∞ is not a branch pt.

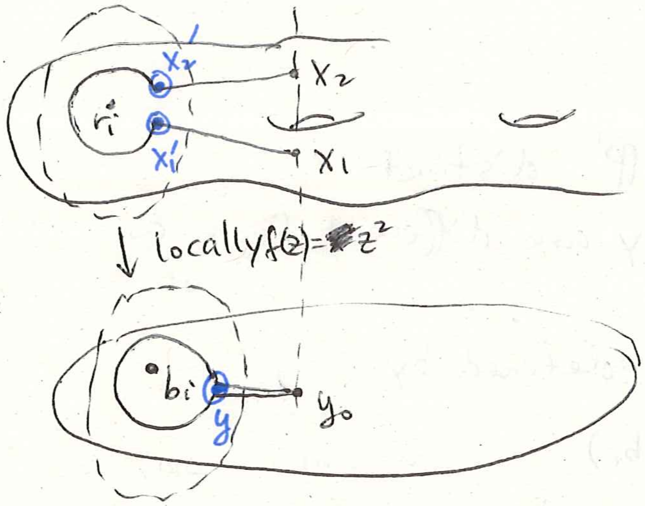
- $|\mathrm{Aut}(p)| = 2$ because an auto. is either Id_X or it switches the two fiber points. (by continuity, it switches one $\Rightarrow$ switches all)

- p is the only hyperell. cover.

<u>Lem</u>: $f : X \to \mathbb{P}^1$ hyperell Hurwitz cover w/ rami locus $R = \{r_1, \dots, r_{2g+2}\} \subset X$ and branch locus $B = \{b_1, \dots, b_{2g+2}\}$, $b_i = f(r_i)$

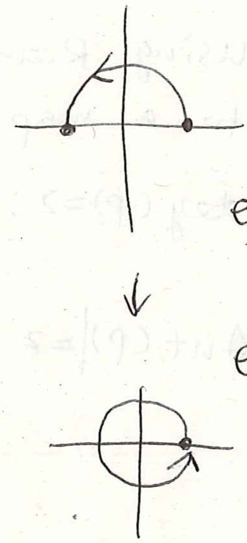
Let $y_0 \in \mathbb{P}^1 \setminus B$ and $f^{-1}(y_0) = \{x_1, x_2\}$

1. If γ is a simple loop based at y_0 which separates $\mathbb{P}^1$ into two regions each containing even # of branch pts. then γ lifts to a loop via f.
2. If γ ------- ------ containing odd # of branch pts. then γ lifts to an open path.



Around a rami. pt., local equation is $f(z)=z^2$

A loop lifts to an open path. (Say, the loop based on a pt $y \in \mathbb{P}^1$)



Then pick a path from y to y_0. ~~The~~ Its lifting to paths from $\{x_1, x_2\} = f^{-1}(y)$ must be disjoint. Thus a loop "$y_0 \leftrightarrows y \circlearrowleft$" lifts to an open path "$x_1 \to x_1'$, $x_2 \leftarrow x_2'$"

(It's a bit sketchy on this note. The loop notation is not standard. But hopefully the proof is clear.)

(Alternative: consider $\int_\gamma \frac{1}{\prod_{i=1}^{n}(x-b_i)} dx$

- $0 \Rightarrow \sqrt{\prod_{i=1}^{n}(x-b_i)}$ can be defined along the path
- nonzero $\Rightarrow \sqrt{\prod_{i=1}^{n}(x-b_i)}$ cannot be defined, need to go through a different sheet of R.S.)

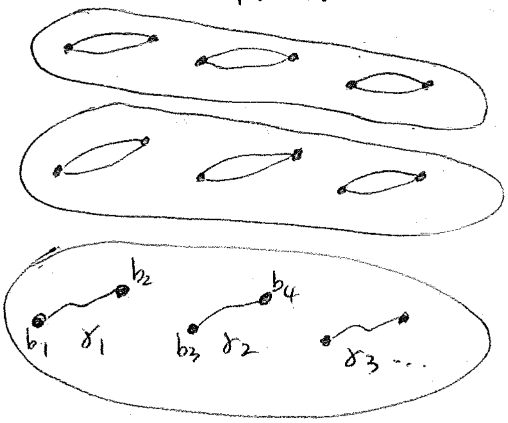
Extend the argument to simple loop containing multiple branch pts. ......

□

Lem: ~~Let f: X → ℙ¹~~

Let $B = \{b_1, \cdots, b_{2g+2}\} \subset \mathbb{P}^1$ be distinct pts.
There exists a unique rami. cover of $\mathbb{P}^1$ of deg 2 with B as its branch pts.

pf. It suffices to prove that there is a unique ram. cover up to homeomorphism. (Because Riemann existence thm)



- $Y := \mathbb{P}^1 \setminus \bigcup_{i=1}^{g+1} \gamma_i$

$f^{-1}(Y) \cong Y \amalg Y$ because there is no nontrivial ~~loop~~ lifting of loop.

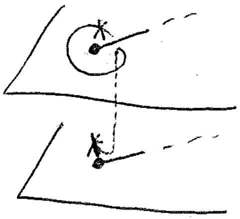
Y is constructed by gluing boundaries of Y
(make Y a mfd with boundary, then quotient topology, $\cdots$)

$\square$

Conclusion: $H_{g \xrightarrow{2} 0}\left((2)^{2g+2}\right) = \frac{1}{2}$

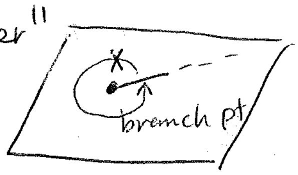
Counting monodromy representation

Cut slits between pairs of branch pts.



When going around a branch pt, the loop might be lifted to paths.

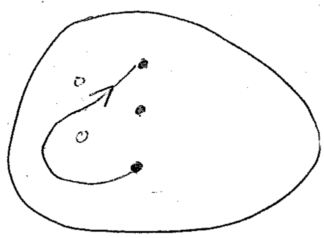
"warp from one sheet to another"



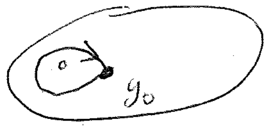
A loop induces an action on points near the ~~branch~~ inverse images of the branch pt.

Let $f: X \to Y$ be a degree d hol. map of conn. R.S. $B = \{b_1, \cdots, b_n\}$

Chose $y_0 \notin B$, consider a loop $\gamma: [0,1] \to Y \setminus B$ based at y_0



Choosing $x \in f^{-1}(y_0)$, γ lifts to a path $\tilde{\gamma}_x$ starting at x. End point is another point in $f^{-1}(y_0)$



$$\sigma_\gamma : f^{-1}(y_0) \longrightarrow f^{-1}(y_0)$$

$$x \longmapsto \tilde{\gamma}_x(1) \quad \left(\begin{array}{l} \tilde{\gamma}_x \text{ lifts } \gamma \text{ w/} \\ \tilde{\gamma}_x(0) = x \end{array} \right)$$

σ_γ is a bijection because all liftings of γ must be disjoint.