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Exercise sheet 1

1. (**Restriction and extension**). Let $k \in \mathbb{N}$ and let $V \subset U \subset \mathbb{R}^d$ be open subsets.

a) Show that there is a natural bounded restriction operator $H^k(U) \to H^k(V)$ defined by $f \mapsto f|_V$ for $f \in H^k(U)$.

b) Show that there is a natural bounded extension operator $H_0^k(V) \to H_0^k(U)$ defined by setting $f \in H_0^k(V)$ equal to zero on $U \setminus V$.

2. (**Existence of weak derivatives**). A function $f \in L^2(\mathbb{T}^d)$ is said to have an α-th (L^2-)*weak derivative* for $\alpha \in \mathbb{N}_0^d$ if there exists $g \in L^2(\mathbb{T}^d)$ such that

$$\int_{\mathbb{T}^d} \psi(x) g(x)\,\mathrm{d}x = (-1)^{\|\alpha\|_1} \int_{\mathbb{T}^d} (\partial_\alpha \psi)(x) f(x)\,\mathrm{d}x$$

holds for any test function $\psi \in C^\infty(\mathbb{T}^d)$. If it exists, such a weak derivative is unique. Show that $f \in H^k(\mathbb{T}^d)$ for some $k \in \mathbb{N}$ if and only if all α-th weak derivatives for $\alpha \in \mathbb{N}_0^d$ with $\|\alpha\|_1 \leq k$ exist.

3. (**An explicit example**). Let $\alpha \in \mathbb{R}$, $U = B_1^{\mathbb{R}^d}(0)$ and consider the function

$$f_\alpha : U \to \mathbb{R},\ x \mapsto \|x\|^\alpha.$$

For which α is $f_\alpha \in H^1(U)$?

HINT: To obtain an elementary solution, you can choose the following approach. Prove first that if f is a continuous function on $[0,1)$ and is continuously differentiable on $(0,1)$ with $f' \in L^2((0,1))$ then $f \in H^1((0,1))$. For this you may use Exercise 5 of the current sheet to approximate f' by a smooth function g and then integrate. See also Exercise 5a), Sheet 13 of FA I or Equation (5.7) in the book on how to compare the L^2-norm of a function and its derivative.

4. (**Existence of derivatives in coordinate directions**). Let $U \subset \mathbb{R}^d$ be open and let $f : U \to \mathbb{R}$ be continuous such that the partial derivatives $\partial_{e_j}^\ell f$ for $j \in \{1, \dots, d\}$ and $\ell \in \mathbb{N}$ exist and are continuous. The goal of this exercise is to show that f is smooth. The technique of the proof will appear again in another context later on.

We first reduce the claim to a *local statement*. For every point $p \in U$ let $\psi_p \in C_c^\infty(U)$ be such that $\psi_p \equiv 1$ on $B_r(p)$ and $\psi_p \equiv 0$ on $B_{2r}(p)$ where $r > 0$ is such that $B_{2r}(p) \subset U$.

a) Show that f is smooth if and only if $\psi_p f$ is smooth for every $p \in U$.

b) Show that the partial $\partial_{e_j}^\ell(\psi_p f)$ for $j \in \{1, \dots, d\}$ and $\ell \in \mathbb{N}$ exist and are continuous.

Thanks to a) and b) above we may assume that f has compact support (in U). We now *transfer the question to the torus*. Let $R > 0$ be such that $\operatorname{supp}(f) \subset (-R/2, R/2)^d$ and note that we can view the latter set as a subset of $\mathbb{R}^d/R\mathbb{Z}^d$. By extending f trivially we obtain a function $\tilde{f}$ on $\mathbb{R}^d/R\mathbb{Z}^d$.

c) Show that f is smooth by applying the corresponding statement on the torus to the function $\tilde{f}$ (cf. Sheet 6 of FA I).

HINT: If you are annoyed by the factor R, you may reduce further to the case where f has compact support in $B_{\frac{1}{2}}(0)$.

5. (**Smooth approximate identites**). The goal of this exercise is to show that L^p-functions for $p < \infty$ can be approximated by smooth functions. For this, we will use the convolution of functions with "highly localized bump functions".

a) Show that the function $\psi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined by

$$\psi(t) = \begin{cases} \mathrm{e}^{\frac{1}{t}} & \text{for } t < 0 \\ 0 & \text{for } t \geq 0 \end{cases}$$

for $t \in \mathbb{R}$ is smooth.

b) Define the function $j : \mathbb{R}^d \to \mathbb{R},\ x \mapsto c\psi(\|x\|_2^2 - 1)$ where $c > 0$ is chosen so that $\int_{\mathbb{R}^d} j(x)\,\mathrm{d}x = 1$ and the functions $j_\epsilon : x \mapsto \frac{1}{\varepsilon^d} j(\frac{x}{\varepsilon})$ for $\epsilon > 0$. Show that j_ϵ is smooth and that $\operatorname{supp}(j_\varepsilon) \subset \overline{B_\epsilon(0)}$.

c) Let $f \in C(\mathbb{R}^d)$. Show that the function

$$f_\epsilon : x \in \mathbb{R}^d \mapsto f * j_\varepsilon(x) = \int_{\mathbb{R}^d} f(y) j_\varepsilon(x - y)\,\mathrm{d}y$$

is smooth and that it converges to f as $\varepsilon \searrow 0$ uniformly on compact sets. Verify also that $\operatorname{supp}(f_\varepsilon) \subset \operatorname{supp}(f) + \overline{B_\varepsilon(0)}$.

d) Conclude that $C_c^\infty(U)$ is dense in $L^p(U)$ for any open set $U \subset \mathbb{R}^d$ and $p < \infty$.

HINT: You may use that $C_c(U) \subset L^p(U)$ is dense (Proposition 2.51).

6. (**Weak derivatives and star-shaped subsets**). Suppose that $U \subset \mathbb{R}^d$ is open, bounded and star-shaped with center 0 (in the sense that $\overline{U} \subset \lambda U$ for all $\lambda > 1$). Let $f, f_1, \ldots, f_d \in L^2(U)$ and assume that f_j is the weak e_j-th derivative of f for all $j \in \{1, \ldots, d\}$ (see Definition 5.8). Show that $f \in H^1(U)$.

HINT: There is a hint on page 568 of the book.