D-MATH Functional analysis II Spring 2019
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Exercise sheet 2

1. (**Partial integration on Sobolev spaces**). Let $U \subset \mathbb{R}^d$ be open and $f \in H^k(U)$, $g \in H_0^k(U)$. Show that for all $\alpha \neq 0$ with $\|\alpha\|_1 \leq k$

$$\int_U (\boldsymbol{\partial}_\alpha f) g \,\mathrm{d}x = (-1)^{\|\alpha\|_1} \int_U f(\boldsymbol{\partial}_\alpha g) \,\mathrm{d}x.$$

(Is the same statement true for $g \in H^k(U)$?)

2. (**Extendability**). This exercise continues in some sense Exercise 1 from Sheet 1. Let $U \subset \mathbb{R}^d$ be open, let $V \subset U$ be an open subset and let $k \in \mathbb{N}$. Show that one cannot define in general an extension operator $H^k(V) \to H^k(U)$ by extending functions to be zero on the complement $U \setminus V$.

HINT: Think of the case $d = 1$, $k = 1$ and constant functions.

3. (**Smooth approximate characteristic function**). Let $U \subset \mathbb{R}^d$ be an open set and let $K \subset U$ be compact. The goal of this exercise is to prove a smooth version of Urysohn's lemma: we claim that there exists some $\phi \in C_c^\infty(U)$ with $\phi|_K \equiv 1$ and with $0 \leq \phi \leq 1$. Proceed by the following steps to find such a function.

a) In Exercise 5 of Sheet 1 we showed that $\psi_1 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined by $\psi_1(t) = 0$ for $t \geq 0$ and $\psi_1(t) = \mathrm{e}^{1/t}$ for $t < 0$ is smooth. Show that

$$\psi_2 : t \in \mathbb{R} \mapsto \frac{\psi_1(t)}{\psi_1(t) + \psi_1(-1-t)} \in [0, 1]$$

is smooth and satisfies $\psi_2(t) = 0$ for $t > 0$ and $\psi_2(t) = 1$ for $t \leq -1$.

b) Show that the function $\psi_3 : t \in \mathbb{R} \mapsto \psi_2(t-2)\psi_2(-2-t)$ is smooth and satisfies $\psi_3(t) = 0$ for $t \notin [-2, 2]$ and $\psi_3(t) = 1$ for $t \in [-1, 1]$.

c) For a point $x \in K$ find a function $\psi_x \in C_c^\infty(U)$ which is equal to 1 in a neighborhood of x.

d) Prove the claim of the exercise.

HINT: Find first a function in $C_c^\infty(U)$ which is positive on K and then compose with an appropriate version of one of the functions ψ_1, ψ_2, ψ_3.

4. (**Optimality of vanishing in square-mean sense**). We proved in Proposition 5.33 that the L^2-norm of traces of an H_0^1-function decays towards the boundary (at least) like $\sqrt{\delta}$ where δ is (roughly) the distance to the boundary. The aim of this exercise is to show that one cannot improve upon the exponent in δ.

a) Let $U \subset \mathbb{R}^d$ be open and let $f \in H^k(U)$ be compactly supported[1] in U. Show that $f \in H_0^k(U)$.

HINT: Choose $\phi \in C_c^\infty(U)$ as in Exercise 3 for $K = \mathrm{supp}(f)$ and multiply with it. Then use a lemma from the lecture. If you have difficulties here, proceed first to the other parts of the exercise.

b) Let $U = B_1(0) \subset \mathbb{R}^2$ and $\alpha > \frac{1}{2}$. Show that the (tent-like) function

$$f : x \in U \mapsto (1 - \|x\|)^\alpha$$

is in $H_0^1(U)$.

HINT: Since U is star-shaped, Exercise 6 in Sheet 1 applies and it suffices to show that f and its first derivates are in $L^2(U)$ in order to verify that $f \in H^1(U)$. To show that $f \in H_0^1(U)$ one can consider for $\lambda > 1$ the functions f_λ (adapted tents) defined as $f_\lambda(x) = f(\lambda x)$ if $\|x\| < \frac{1}{\lambda}$ and $f_\lambda(x) = 0$ otherwise.

c) Compute the norms[2] $\|f|_{\partial B_{1-\delta}(0)}\|_{L^2}$ for $\delta \in (0, 1)$ and explain how this relates to the goal formulated at the beginning of the exercise.

5. (**Difference of Sobolev spaces**).

a) Let $U \subset \mathbb{R}^d$ be a non-empty bounded open set. Show that $H_0^k(U)$ is a proper subspace of $H^k(U)$.

HINT: Look at constant functions and Exercise 1. If U has a nice boundary, there is a simpler argument.

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[1] Formally, we require f to have a representative with compact support in U (as f is only defined up to null-sets).

[2] The measure we consider on these circles is simply the probability measure defined in analogy to the torus.

b) Show that $H^k(\mathbb{R}^d) = H_0^k(\mathbb{R}^d)$.

HINT: Given a function $f \in C^\infty(\mathbb{R}^d) \cap H^k(\mathbb{R}^d)$ choose a large ball outside of which f and all its weak derivatives have L^2-norm $< \varepsilon$. Then consider ψf for a suitable $\psi \in C_c^\infty(\mathbb{R}^d)$ which is equal to one on that ball – for this look at Exercise 3c) above.

6. (**An extension operator**). We know from Exercise 2 that extension by zero is not typically well-defined on Sobolev spaces. There are however reasonable extensions; we illustrate this in an example.

a) Show that the set of functions $\psi \in C^1(\overline{\mathbb{R}_{>0}}) \cap H^1(\mathbb{R}_{>0})$ with $\psi'(0) = 0$ is dense in $H^1(\mathbb{R}_{>0})$.

HINT: For $f \in C^\infty(\mathbb{R}_{>0}) \cap H^1(\mathbb{R}_{>0})$ consider a point $x_0 > 0$ very close to 0 and replace the trajectory of f between 0 and x_0 appropriately.

b) Show that there is a bounded extension operator

$$\mathrm{ext} : H^1(\mathbb{R}_{>0}) \to H^1(\mathbb{R})$$

with the property that for all $f \in H^1(\mathbb{R}_{>0})$ we have that $\mathrm{ext}(f)|_{\mathbb{R}_{>0}} = f$.

HINT: Consider the reflection around zero.