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Exercise sheet 3

1. (**Weakly harmonic functions**). Let $U \subset \mathbb{R}^d$ be an open subset. Show that for any $f \in H^1(U)$ and $\phi \in C_c^\infty(U)$ we have

$$\langle f, \phi \rangle_1 = -\langle f, \triangle\phi \rangle_{L^2(U)}.$$

Recall that we say that $f \in H^1(U)$ is *weakly harmonic* if this expression vanishes for all ϕ.

2. (**A decomposition of H^1-functions**). Let $U \subset \mathbb{R}^d$ be an open and bounded subset.

a) Show that any weakly harmonic function $f \in H_0^1(U)$ is zero.

b) Let $f \in H^1(U)$. Show that there is a *unique* decomposition $f = g + v$ where $g \in H^1(U)$ is weakly harmonic and $v \in H_0^1(U)$.

3. (**Smooth partition of unity**). Let $K \subset \mathbb{R}^d$ be compact. Let $\mathcal{O}$ be a cover of K with open subsets $O \subset \mathbb{R}^d$. The aim of this exercise is to show that there are finitely many functions $\chi_1, \ldots, \chi_k \in C_c^\infty(\mathbb{R}^d)$ with

(i) $\sum_{i=1}^k \chi_i$ is equal to 1 on K and
(ii) for every i we have $\mathrm{supp}(\chi_i) \subset O$ for some $O \in \mathcal{O}$.

Such a collection of functions is called a *smooth partition of unity* for the cover $\mathcal{O}$. To prove the existence proceed as follows:

a) Find $r > 0$ such that for any $x \in K$ the ball $B_{2r}(x)$ is contained in some element of the open cover $\mathcal{O}$.

b) For any $x \in K$ find $\chi \in C_c^\infty(B_{2r}(x))$ with $\chi|_{B_r(x)} = 1$.

c) Take a finite open cover of K with balls of radius r and conclude.

HINT: See also the functions constructed in Exercise 3, Sheet 2.

4. (**Diffeomorphisms and Sobolev spaces**). Let $U \subset \mathbb{R}^d$ be open and bounded and let $\tilde{U}$ be an open set containing $\overline{U}$. Let $\Phi : \tilde{U} \to \tilde{V} \subset \mathbb{R}^d$ be a diffeomorphism and let $V = \Phi(U)$. We claim that Φ induces an isomorphism $H^k(V) \to H^k(U)$ by precomposition.

a) Let $f \in C^\infty(V) \cap H^k(V)$ and let α be with $\|\alpha\|_1 \leq k$. Show inductively that $\partial_\alpha(f \circ \Phi)$ is of the form $\sum_{\|\beta\|_1 \leq \|\alpha\|_1} g_{\alpha,\beta} \cdot (\partial_\beta f) \circ \Phi$ for smooth functions $g_{\alpha,\beta}$ on $\tilde{V}$ which do not depend on f.

b) Show that Φ induces a (well-defined) isomorphism

$$f \in H^k(V) \mapsto f \circ \Phi \in H^k(U). \tag{1}$$

Assume now that Φ is given by an isometry, that is, there is an orthogonal matrix $R \in \mathrm{O}(d)$ and a dilation $a \in \mathbb{R}^d$ so $\Phi(x) = Rx + a$ for all $x \in \tilde{U}$. Show that the map in (1) is an isometry.

HINT: For the last claim one can study the argument for a) with more care and explicitly compute the factors.

This exercise indicates that there is a definition of a Sobolev space for (Riemannian) manifolds which does not depend on the choice of charts.

5. (**Sobolev embedding theorem and uniformity on compacta**). Let $U \subset \mathbb{R}^d$ be open, let $K \subset U$ be compact and let $k > \frac{d}{2}$. Show that

$$\|f|_K\|_\infty \ll_K \|f\|_{H^K(U)}$$

for every $f \in H^k(U)$.

6. (**Poisson integral representation formula**). The aim of this exercise is to illustrate that the Dirichlet boundary value problem can be solved explicitly in special cases. We consider here the open subset $\mathbb{H}^2 = \{x + \mathrm{i}y \in \mathbb{C} : y > 0\}$ and let $f \in C_b(\partial\mathbb{H}^2)$ (where $\partial\mathbb{H}^2 = \mathbb{R} \subset \mathbb{C}$ is the real line). Define for $x + \mathrm{i}y \in \mathbb{H}^2$

$$\tilde{f}(x+iy) = \tfrac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} f(t)\,\mathrm{d}t.$$

a) Show that $\tilde{f}$ is well-defined (i.e. that the above Riemann integral is convergent) and continuous.

b) Let $\chi : \tau \in \mathbb{R} \mapsto \frac{1}{\pi}\frac{1}{1+\tau^2}$ and $\chi_y : \tau \mapsto \frac{1}{y}\chi(\frac{\tau}{y})$. Show that for any $x + \mathrm{i}y \in \mathbb{H}^2$

$$\tilde{f}(x+iy) = (\chi_y * f)(x).$$

Conclude from this that $\tilde{f}$ extends continuously to $\partial\mathbb{H}^2$ where it is equal to f.

c) Show that $\tilde{f}$ is harmonic and bounded.