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Exercise sheet 4

1. (**H_0^1 and absolute convergence**). Let $U \subset \mathbb{R}^d$ be open and bounded and denote by $f_1, f_2, \ldots$ in $H_0^1(U) \cap C^\infty(U)$ a sequence of eigenfunctions of the Laplace operator on U with eigenvalues $\lambda_1, \lambda_2, \ldots$ (cf. Theorem 6.56) which forms an orthonormal basis of $L^2(U)$. Furthermore, let $g = \sum_{n=1}^\infty a_n f_n \in L^2(U)$. Show that $g \in H_0^1(U)$ if and only if

$$\sum_{n=1}^\infty |a_n|^2 |\lambda_n| < \infty.$$

HINT: Use Lemma 6.67.

2. (**Growth of Laplacian eigenvalues**). Let $U \subset \mathbb{R}^d$ be open, bounded and Jordan-measurable. Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of $\triangle$ on U (with multiplicities) ordered such that $0 > \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$. Show that

$$\lim_{n \to \infty} \frac{|\lambda_n|}{n^{2/d}} = (2\pi)^2 (\omega_d m(U))^{-2/d}.$$

HINT: Choose the right sequence of T's in Weyl's law. To take care of multiplicities in eigenvalues, you might also want to adapt Weyl's law to treat the counting function $T \mapsto |\{m : |\lambda_m| < T\}|$.

3. (**Eigenfunctions of the Laplacian on a rectangle**). Consider the open (rectangular) set $U = (0, a_1) \times \ldots \times (0, a_d) \subset \mathbb{R}^d$. The aim of this exercise is to find a complete set of explicit eigenfunctions of the Laplacian on U.

a) Show that a function of the form

$$f : x \in \mathbb{R}^d \mapsto \sin(\lambda_1 x) \cdots \sin(\lambda_d x)$$

is a smooth eigenfunction of the Laplace operator and characterize the (countable) set of tuples $(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d_{>0}$ for which f vanishes identically on the boundary of U.

REMARK: One can use the same idea as in Exercise 4b), Sheet 2 to show that $f \in H_0^1(U)$.

b) Show that the functions from a) form an orthogonal basis of $L^2(U)$ and explain what this implies in the context of Theorem 6.56.

4. (**An error rate in Weyl's law**). Let $U \subset \mathbb{R}^d$ be open, bounded and Jordan-measurable. Then Weyl's law as in Theorem 6.64 states that the eigenvalue counting function $N_U(T) = |\{n : |\lambda_n| \leq T\}|$ satisfies

$$N_U(T) = C_U T^{\frac{d}{2}} + o(T^{\frac{d}{2}})$$

for an explicit constant C_U depending on U. The aim of this exercise is to understand the error $E_U(T) = |N_U(T) - C_U T^{\frac{d}{2}}|$. For simplicity[1] we replace U by the torus $\mathbb{T}^d$.

a) Show that a bound of the kind

$$\left| |\mathbb{Z}^d \cap \overline{B_S^{\mathbb{R}^d}(0)}| - \mathrm{vol}(B_S^{\mathbb{R}^d}(0)) \right| \ll S^a \tag{1}$$

for all $S > 0$ implies that $E_{\mathbb{T}^d}(T) = O(T^{\frac{a}{2}})$.

b) Verify the bound in (1) for $a = d - 1$.

HINT: Revisit the proof of Proposition 6.65. The problem of finding good exponents a as in (1) for $d = 2$ is called the Gauss circle problem. Conjecturally, the error should be bounded by $c_\varepsilon S^{\frac{1}{2}+\varepsilon}$ for any $\varepsilon > 0$. The current record $a = \frac{517}{824} + \varepsilon = 0.627\ldots + \varepsilon$ is due to Bourgain and Watt.

5. (**Supremum bounds for eigenfunctions on compacta**). Let $U \subset \mathbb{R}^d$ be open and bounded and let $f \in H_0^1(U) \cap C^\infty(U)$ be an eigenfunction of the Laplace operator[2] for eigenvalue $\lambda < 0$. We want to show that for any compact subset $K \subset U$

$$\|f\|_{K,\infty} = \|f|_K\|_\infty \ll_{K,U} |\lambda|^{\frac{d}{4}+\frac{1}{2}} \|f\|_{L^2}.$$

This estimate controls the *growth* of the L^2-normalized Laplacian eigenfunctions on compacta in terms of their eigenvalues. We proceed in steps.

a) Adapt the proof of Lemma 5.48 to show that for any $g \in L^2(\mathbb{T}^d)$ with $\Delta g = u \in H^k(\mathbb{T}^d)$ we have $g \in H^{k+2}(\mathbb{T}^d)$ and

$$\|g\|^2_{H^{k+2}(\mathbb{T}^d)} \ll_k \|g\|^2_{L^2(\mathbb{T}^d)} + \|u\|^2_{H^k(\mathbb{T}^d)}.$$

b) Let f be as in the beginning of the exercise. Revisit the proof of Theorem 5.45 to show that for any $\chi \in C_c^\infty(U)$

$$\|\chi f\|_{H^k(U)} \ll_{k,\chi} |\lambda|^{\frac{k}{2}} \|f\|_{L^2(U)}.$$

HINT: If suffices to consider the case $|\lambda| > 1$.

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[1] of the exposition but not necessarily of the result

[2] This implies that it is an eigenfunction of $-\iota\iota^*$ where $\iota : H_0^1(U) \to L^2(U)$ is the inclusion.

c) Use Exercise 5 on Sheet 3 to deduce the desired statement.

6. (**The heat equation**). Let $U \subset \mathbb{R}^d$ be an open and bounded subset with smooth boundary. Let $u_0 \in L^2(U)$ be a given initial heat distribution. We would like to analyze the *heat equation*

$$\partial_t u = \triangle_x u$$

with boundary values

$$\begin{cases} u(x,t) = 0 \text{ for all } x \in \partial U \text{ and } t > 0 \\ u(x,0) = u_0(x) \text{ for all } x \in U. \end{cases}$$

Here, u is a function in the position $x \in U$ and in time $t \in \mathbb{R}_{\geq 0}$. Also, $\triangle_x$ denotes the Laplacian taken only in the position.

a) Read Section 1.2.1 (*principle of superposition*) and explain why one should attempt to solve the heat equation (with boundary values) by the ansatz

$$u(x,t) = \sum_{n=1}^{\infty} a_n f_n(x) \mathrm{e}^{\lambda_n t}. \tag{2}$$

Here, the functions $f_n \in C^\infty(U) \cap H_0^1(U)$ form an orthonormal basis of $L^2(U)$ consisting of eigenfunctions of the Laplace operator on U with eigenvalues λ_n. Furthermore, the coefficients a_n are chosen such that $u_0 = \sum_{n=1}^{\infty} a_n f_n$.

In the following we want to make this ansatz more precise. So let u be as in (2).

b) Show that $u(\cdot,t) \to u_0$ in $L^2(U)$ and that $u(\cdot,t) \in H_0^1(U)$ for any $t > 0$. In this sense, the boundary constraints are satisfied.

HINT: For the latter you can use Exercises 1 and 2.

c) Let $K \subset U$ be compact. Use Exercises 2 and 5 to show that the series in (2) converges uniformly on K for any fixed $t > 0$. Deduce that $u(\cdot,t)$ is continuous on U and (using your proof) that

$$\|u(\cdot,t)\|_{K,\infty} \to 0 \text{ as } t \to \infty.$$

d) State an estimate for the derivatives of f_n that you suspect to hold in analogy to Exercise 5. Use it to prove that $u \in C^\infty(U \times \mathbb{R}_{>0})$.