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Exercise sheet 5

1. (**More on Theorem 7.54**). Let X be a locally compact σ-compact metric space and let $\Lambda \in C_0(X)^*$. By Theorem 7.54 there exists a positive finite measure $|\mu|$ on (Borel sets of) X and some measurable[1] g with $\|g\|_\infty = 1$ such that

$$\Lambda(f) = \int_X fg \,\mathrm{d}|\mu|$$

for all $f \in C_0(X)$. Furthermore, we know that $|\mu|(X) = \|\Lambda\|_{\mathrm{op}} = \|g\|_{L^1_{|\mu|}(X)}$. Show that then $|g(x)| = 1$ for $|\mu|$-almost every $x \in X$.

HINT: For a contradiction you may suppose that there is a set of positive measure on which $|g| < 1-\varepsilon$ for some $\varepsilon > 0$. You may restrict yourself to the real case as in the lecture.

2. (**Rotation on the circle and Koopman eigenvalues**). Let $\alpha \in \mathbb{T}$ be fixed and consider the rotation $R_\alpha : x \in \mathbb{T} \mapsto x + \alpha \in \mathbb{T}$.

a) Prove that T induces a unitary operator

$$U_\alpha : L^2(\mathbb{T}) \to L^2(\mathbb{T}),\ f \mapsto f \circ R_\alpha.$$

b) Characterize the set of α's for which U_α has a non-zero eigenvector with eigenvalue 1.

c) Show that (independently of the choice of α) there is an orthonormal basis of $L^2(\mathbb{T})$ consisting eigenvectors of U_α. We say in this case that U_α has *purely discrete spectrum.*

3. (**Signed measures**). In this exercise we review the notion of signed measures and prove *polar decompositions*. Let X be a locally compact σ-compact metric space. Recall that a *real signed measure* μ on X is by definition of the form $\mu = \mu_1 - \mu_2$ for positive finite measure μ_1, μ_2 on X.

[1] Real- or complex valued as the choice of field is implicit in $C_0(X)$.

a) Show that every real signed measure μ can be written as $\mu_1 - \mu_2$ as above with the additional property $\mu_1 \perp \mu_2$ by either applying Lebesgue decomposition or Theorem 7.54. Verify using this the formulas

$$\mu_1(A) = \sup\{\mu(B) : B \subset A \text{ measurable}\},$$
$$\mu_2(A) = -\inf\{\mu(B) : B \subset A \text{ measurable}\}.$$

Recall that a *complex signed measure* μ on X is of the form $\mu = \mu_{\text{real}} + \mathrm{i}\mu_{\text{imag}}$ for real signed measures $\mu_{\text{real}}, \mu_{\text{imag}}$ on X (by definition).

b) Show that for any complex signed measure μ on X one can find a positive finite measure $|\mu|$ on X and $h : X \to \mathbb{C}$ measurable with $|h(x)| = 1$ for $|\mu|$-almost every $x \in X$ such that $\mathrm{d}\mu = h\,\mathrm{d}|\mu|$.

HINT: You may apply Theorem 7.54 (the real case as proven in the lecture).

4. (**Translation on a non-compact group**). Consider the locally compact abelian group $G = \mathbb{Z} \times \mathbb{T}$ who is equipped with a Haar measure m given by the product measure of the counting measure on $\mathbb{Z}$ and the Haar (Lebesgue) measure on $\mathbb{T}$. Define for any $g \in G$ the translation $T_g : h \in G \mapsto h + g \in G$ as well as the induced unitary operator

$$U_g : L^2_m(G) \to L^2_m(G),\ f \mapsto f \circ T_g.$$

a) Characterize the set of $g \in G$ for which U_g has does not have an eigenvector (for any eigenvalue). In this case we say that U_g has *purely continuous spectrum*.

b) Characterize the set of $g \in G$ for which U_g has

- purely discrete spectrum or
- *mixed spectrum* (that is, there are eigenvectors of U_g but there is no orthonormal basis consisting of eigenvectors).

5. (**Non-reflexivity of** $C([0,1])$). Consider the compact metric space $X = [0,1]$. We denote by $\mathscr{L}^\infty(X)$ the vector space of bounded measurable functions on X and by $\mathcal{M}(X)$ the vector space of real (or complex) signed measures. One can equip $\mathcal{M}(X)$ with a natural norm (cf. Exercise 3.33) for which it is then a Banach space.

a) Notice that every signed measure μ on X yields a linear functional on $\mathscr{L}^\infty(X)$ via $f \mapsto \int_X f\,\mathrm{d}\mu$. Show that not every linear functional on $\mathscr{L}^\infty(X)$ is of this form.

b) Note that conversely any $f \in \mathscr{L}^\infty(X)$ yields a bounded linear functional on $\mathcal{M}(X) \cong C(X)^*$. Deduce that $C(X)$ is not reflexive.

BONUS: Show that $\mathcal{M}(X)^*$ contains more functionals than those who arise from $\mathscr{L}^\infty(X)$.

HINT: This is Exercise 7.58 in the book for which a hint may be found on page 573.

6. (**Difference quotients and L^2-convergence**). The aim of this exercise is to give a flavour of the techniques involved for elliptic regularity at the boundary (cf. Section 8.2.2). For $f \in L^2(\mathbb{R})$ and $h \in \mathbb{R}$ non-zero we define

$$D^h f(x) = \frac{f(x+h) - f(x)}{h}$$

for almost every $x \in \mathbb{R}$.

a) Show that for any $f \in C_c^\infty(\mathbb{R})$ we have $D^h f \to f'$ as $h \to 0$ in $L^2(\mathbb{R})$.

HINT: Use either Taylor's theorem or the mean value theorem to show uniform convergence.

We now assume that $f \in L^2(\mathbb{R})$ is a function such that $\|D^h f\|_{L^2(\mathbb{R})} \leq M$ for some fixed number $M > 0$.

b) Prove that f has a weak derivative $f' \in L^2(\mathbb{R})$ and that $D^h f \to f'$ as $h \to 0$ in the *weak* topology on $L^2(\mathbb{R})$ where f' is the weak derivative of f.

HINT: Use the Banach-Alaoglu Theorem to find a weak limit g along a subsequence. Then verify the formula $\langle D^h f, \phi \rangle = -\langle f, D^{-h}\phi \rangle$ for all $\phi \in C_c^\infty(\mathbb{R})$ and use it together with a) to show that $g = f'$ is the weak derivative of f.

c) Show that for almost every $x, y \in L^2(\mathbb{R})$ we have

$$f(y) - f(x) = \int_x^y f'(t)\,\mathrm{d}t$$

HINT: Apply b) for the inner product with a suitable L^2-function and recall that almost every point of f is a Lebesgue point.

d) Show that $D^h f \to f'$ as $h \to 0$ in the *norm* topology on $L^2(\mathbb{R})$.