D-MATH
Prof. Dr. Manfred Einsiedler
Andreas Wieser

Functional analysis II

Spring 2019

Exercise sheet 6

In this exercise sheet, $\mathcal{H}$ is a separable Hilbert space and $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator. For any $v \in \mathcal{H}$ we denote by $\mathcal{H}_v$ the cyclic subspace generated by v and by μ_v the *spectral measure* associated to v (cf. Corollary 9.8). It is a finite measure on $\mathbb{T}$ uniquely determined by the property

$$\langle U^n v, v\rangle = \int_{\mathbb{T}} \chi_n \, \mathrm{d}\mu_v$$

for every $n \in \mathbb{Z}$. Using this measure one then obtains a unitary isomorphism $\mathcal{H}_v \cong L^2_{\mu_v}(\mathbb{T})$ where the action of U on $\mathcal{H}_v$ corresponds to the action of M_{χ_1} on $L^2_{\mu_v}(\mathbb{T})$ and where the vector v corresponds to $\mathbb{1} \in L^2_{\mu_v}(\mathbb{T})$.

1. (**Mixed vectors**). Let $v, w \in \mathcal{H}$. Show that there exists a complex signed measure $\mu_{v,w}$ with

$$\langle U^n v, w\rangle = \int_{\mathbb{T}} \chi_n \, \mathrm{d}\mu_{v,w}$$

for all $n \in \mathbb{Z}$.

HINT: Set $\mu_{v,w} = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \mu_{v+\mathrm{i}^k w}$.

2. (**Eigenfunctions and spectral measures**). Let $\lambda \in \mathbb{T}$ and $v \in \mathcal{H}$ be non-zero. Show that v is an eigenvector of U for eigenvalue $\mathrm{e}^{2\pi\mathrm{i}\lambda}$ if and only if μ_v is equal to $\|v\|^2$ times the Dirac measure at the point λ.

3. (**Wiener's lemma**). Let μ be a finite measure on the torus $\mathbb{T}$ (e.g. a spectral measure) and denote by $p_n(\mu)$ the Fourier coefficients of the measure μ as in Example 9.4. Roughly speaking, Wiener's lemma establishes a relation between the Fourier coefficients and the atoms of the measure μ.

a) Recall that a point $x_0 \in \mathbb{T}$ is an *atom* of μ if $\mu(\{x_0\}) > 0$. Show that μ has at most countably many atoms. We denote these by $x_1, x_2, \ldots$.

b) Show that for any finite measure ν on $\mathbb{T}$ we have

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} p_n(\nu) = \nu(\{0\}).$$

HINT: See Definition 3.61 ("Dirichlet kernel") and the lemma thereafter.

c) Consider the product measure $\mu \times \mu$ on $\mathbb{T}^2$ and denote by $\triangle = \{(t,t) : t \in \mathbb{T}\}$ the diagonal. Let ν be the push-forward of $\mu \times \mu$ under the map $(t_1, t_2) \in \mathbb{T} \mapsto t_1 - t_2 \in \mathbb{T}$. Show that $\nu(\{0\}) = \mu \times \mu(\triangle)$ and that $p_n(\nu) = |p_n(\mu)|^2$.

d) Show that $\mu \times \mu(\triangle) = \sum_{j=1}^{\infty} \mu(\{x_j\})^2$ and deduce Wiener's lemma:

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |p_n(\mu)|^2 = \sum_{j=1}^{\infty} \mu(\{x_j\})^2.$$

4. (**Explicit examples**).

a) (*Shift operator*) Consider the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$ and the unitary operator $U : a = (a_n)_n \in \mathcal{H} \mapsto (a_{n-1})_n$. Let $v \in \ell^2(\mathbb{Z})$ be the sequence which satisfies $v_0 = 1$ and $v_n = 0$ for $n \neq 0$. Show that $\mathcal{H} = \mathcal{H}_v$ and compute the spectral measure μ_v.

b) (*A toral automorphism*) Consider a toral automorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$ induced by an invertible integer matrix A,

$$T(x + \mathbb{Z}^2) = Ax + \mathbb{Z}^2.$$

If $|\det A| = 1$ is then T is measure preserving with respect to the Lebesgue measure m on $\mathbb{T}^2$ and there is an associated unitary operator U_T on $\mathcal{H} = L^2(\mathbb{T}^2)$ defined by precomposing a function with T. Compute for any $n \in \mathbb{Z}^2$ the spectral measure μ_{χ_n} of the character $\chi_n \in L^2(\mathbb{T}^2)$ under the additional assumption that A is diagonalizable over $\mathbb{R}$.

5. (**Spectral theorem for commuting operators**). In this exercise we generalize the discussions of Theorem 9.2, 9.6 to unitary representations of $\mathbb{Z}^d$ on $\mathcal{H}$.

a) Define the notion of a positive definite functions $p \in \mathbb{Z}^d \mapsto \mathbb{C}$ and generalize Herglotz' theorem to such functions (where $d = 1$ should correspond to the already proven case).

b) Let $\pi : \mathbb{Z}^d \to B(\mathcal{H})$ be a unitary representation of $\mathbb{Z}^d$ on $\mathcal{H}$. Generalize Corollary 9.8 (and thus also Theorem 9.2) to such representations (the action of d commuting unitary operators).

6. (**Polynomial recurrence**). Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be invertible and measure-preserving. The goal of this exercise is to prove the following theorem refining the Poincare recurrence theorem[1]:

> **Theorem:** *Let $A \in \mathcal{B}$ with $\mu(A) > 0$ and let $p \in \mathbb{Z}[x]$ with zero constant coefficient. Then there exists $n \in \mathbb{N}$ such that $\mu(T^{-p(n)}A \cap A) > 0$.*

To attack this theorem using spectral theory, we consider the (Koopman) unitary operator $U : L^2(X, \mu) \to L^2(X, \mu), f \mapsto f \circ T$. The mean ergodic theorem (FA I, Sheet5) states that $\frac{1}{N}\sum_{n=1}^{N} U^n f$ converges to an eigenfunction of U of eigenvalue 1 (in this case a T-invariant L^2-function). For $\chi(\vartheta) = \exp 2\pi i\vartheta$, let us denote the eigenspaces

$$L^2_\vartheta = \{f \in L^2(X, \mu) : Uf = \chi(\vartheta)f\} \quad \text{and} \quad L^2_{\text{rat}} = \bigoplus_{\vartheta \in \mathbb{Q}} L^2_\vartheta.$$

Note that the constant function belongs to L^2_{rat}.

a) (*Reformulation in terms of the spectral language*) Show that polynomial recurrence follows if one can establish the following implication:

$$f \in L^2(X, \mu) \text{ such that } \langle U^{p(n)} f, f\rangle_{L^2(X,\mu)} = 0 \text{ for all } n \in \mathbb{N}_{>0} \Rightarrow f \in (L^2_{\text{rat}})^\perp.$$

b) Let $f \in L^2(X, \mu)$. Show that you can write

$$\left\| \frac{1}{N} \sum_{n=1}^{N} U^{p(n)} f \right\|^2 = \int_0^1 \left| \frac{1}{N} \sum_{n=1}^{N} \chi(p(n)\vartheta) \right|^2 d\nu_f(\vartheta)$$

for some finite measure ν_f on the torus.

c) Weyl's equidistribution theorem[2] states that $\exp 2\pi i p(n)\vartheta$ is equidistributed on the torus for irrational ϑ (compare this to FA I, Sheet 6, Exercise 5). Using this fact, show that $\nu_f(\mathbb{Q}/\mathbb{Z}) = 0$ implies $\frac{1}{N}\sum_{n=1}^{N} U^{p(n)} f \to 0$ in L^2 for all $f \in L^2(X, \mu)$.

d) Show that $\nu_f(\mathbb{Q}/\mathbb{Z}) = 0$ for all $f \in (L^2_{\text{rat}})^\perp$.

HINT: Suppose that there is $\phi = \frac{b}{a} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ with $\nu_f(\{\phi\}) > 0$. Consider the characteristic function $\mathbb{1}_{\{\phi\}}$ and show that there exists $g \in L^2_{\text{rat}}$ such that $\langle g, f\rangle_\mu = \langle \mathbb{1}_{\mathbb{T}}, \mathbb{1}_{\{\phi\}}\rangle_{\nu_f}$.

e) Let $f \in L^2_{\text{rat}}$. Prove that for any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ we have $\|U^{mn} f - f\| < \varepsilon$.

f) Decompose $f = f_{\text{rat}} + f_{\text{rat}}^\perp$ and prove the reformulation of a) combining the statement proved in c) and e) (applied to the polynomial $x \mapsto p(mx)$).

[1] See Theorem 2.11 in *Ergodic Theory with a view towards Number Theory* by M. Einsiedler and T. Ward (Springer).

[2] See Theorem 1.4 in *Ergodic Theory with a view towards Number Theory* by M. Einsiedler and T. Ward (Springer).