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Exercise sheet 7

1. (**Uniqueness of barycentres**). Let K be a compact convex metrizable subset of a locally convex vector space. Let μ be a Borel probability measure on K. Show that the barycentre of μ is uniquely determined.

HINT: Assume that there are two barycentres and use Theorem 8.73.

2. (**Equivalent definitions of extreme points**). Let X be a locally convex vector space, let $K \subset X$ be convex and let $x \in K$. Show that the following two properties (definitions of extreme points) are equivalent.

(i) Whenever $x = \frac{y+z}{2}$ for $y, z \in K$ then $y = z = x$.

(ii) Whenever $x = ty + (1-t)z$ with $t \in (0,1)$ and $y, z \in K$ then $y = z = x$.

3. (**Non-existence of extremal points**). Consider the Banach space $L^1_m([0,1])$ where m is the Lebesgue measure. Show that the closure $K = \overline{B_1(0)}$ of the unit ball of $L^1_m([0,1])$ does not have any extremal points.

REMARK: The moral reason for this to be possible is that $L^1_m([0,1])$ is not the dual space of any Banach space. Otherwise, one could use the induced weak*topology (for which K is compact) and the Theorem of Krein-Millman to show the existence of extremal points.

4. (**Carathéodory's form of Minkowski's theorem**). Let $K \subset \mathbb{R}^n$ be a convex compact subset. Show that any $x_0 \in K$ is a convex combination of $(n+1)$ extreme points of K.

HINT: This is Exercise 8.91 with a hint on page 576.

5. (**A closed set of extremal points**). Let K be the closure of the unit ball in the Banach space $\ell^\infty(\mathbb{N})$.

a) Characterize the set of extremal points $\text{ext}(K)$ on K.

HINT: Think first about what would happen if we were to consider $\ell^\infty(I)$ for a finite index set I instead.

b) Show that $\text{ext}(K) \subset K$ is closed.

6. (**Herglotz' theorem vs Choquet's theorem**). The aim of this exercise is to provide another view on Herglotz' theorem and relate it to Choquet's theorem. Let $\mathcal{P}(\mathbb{T})$ be the space of probability measures[1] on the torus $\mathbb{T}$. We also denote by $\mathcal{PD}^1(\mathbb{Z})$ the set of positive definite sequences $(p_n)_{n\in\mathbb{Z}}$ in $\mathbb{C}$ with $p_0 = 1$.

a) Preferably without using Herglotz' theorem show that any $p \in \mathcal{PD}^1(\mathbb{Z})$ is bounded by 1 in absolute value and that $p_{-n} = \overline{p_n}$ for every $n \in \mathbb{Z}$. In particular, we can view $\mathcal{PD}^1(\mathbb{Z})$ as a subset of $\ell^\infty(\mathbb{Z})$.

b) Notice that $\ell^\infty(\mathbb{Z})$ is the dual space $\ell^1(\mathbb{Z})$ via the natural pairing (cf. FAI, Sheet 8). In particular, we obtain a weak*topology on $\ell^\infty(\mathbb{Z})$. Show that $\mathcal{PD}^1(\mathbb{Z})$ is weak*-compact and convex.

c) Show that the map

$$\mathcal{P}(\mathbb{T}) \to \mathcal{PD}^1(\mathbb{Z}), \quad \mu \mapsto (p_n(\mu))_n$$

is continuous and affine.

Note that Herglotz' theorem implies surjectivity of this map and thus the above map is in fact a homeomorphism[2].

d) Show that for every extremal point $p \in \mathcal{PD}^1(\mathbb{Z})$ there is a unique $t_0 \in \mathbb{T}$ such that $p_n = \chi_n(t_0)$ for every $n \in \mathbb{Z}$. Why does this yield a homeomorphism between $\mathbb{T}$ and $\text{ext}(\mathcal{PD}^1(\mathbb{Z}))$?

HINT: You may use Herglotz' theorem here.

e) Explain using d) the meaning of Choquet's theorem in this setting.

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[1] By the Riesz representation theorem we equip $\mathcal{P}(\mathbb{T})$ with the weak*topology. For this topology $\mathcal{P}(\mathbb{T})$ is a compact and metrizable – see Proposition 8.27.

[2] A bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.