D-MATH
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Exercise sheet 8

Throughout this exercise sheet, G is a locally compact, σ-compact metrizable group and m_G is a left Haar measure on G.

1. (**Adjoining a unit**). Let $\mathcal{A}$ be a Banach algebra over $\mathbb{C}$ (possibly non-unital) and define $\mathcal{A}_{\mathbb{1}} = \mathcal{A} \oplus \mathbb{C}\mathbb{1}$ for a symbol $\mathbb{1}$. Then $\mathcal{A}_{\mathbb{1}}$ is a vector space which we equip with

$$\|a + \lambda \mathbb{1}\| = \|a\| + |\lambda|$$

and with a multiplication

$$(a_1 + \lambda_1 \mathbb{1}) \cdot (a_2 + \lambda_2 \mathbb{1}) = (a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1) + \lambda_1 \lambda_2 \mathbb{1},$$

which is bilinear. Show that these structures turn $\mathcal{A}_{\mathbb{1}}$ into a unital Banach algebra with multiplicative unit $\mathbb{1}$.

NOTE: You need to check that the multiplication is associative and that the norm property of Banach algebras is satisfied.

2. (**Examples of Haar measures**). The aim of this exercise is to give to explicit examples of Haar measures.

a) Consider the group $G = \mathrm{SL}_n(\mathbb{R})$, which we view as a closed subset of the metric space $\mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. Let λ be the Lebesgue measure on $\mathrm{Mat}_n(\mathbb{R})$. For $A \subset G$ measurable we define

$$m_G(A) = \lambda\big(\{tg : t \in [0,1], g \in A\}\big).$$

Show that m_G is a left and a right Haar measure on $G = \mathrm{SL}_n(\mathbb{R})$.

b) Consider the $ax + b$-group

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \right\}.$$

Show that in these coordinates the measure m_B given by $\mathrm{d}m_B = \frac{1}{a^2}\,\mathrm{d}a\,\mathrm{d}b$ defines a left Haar measure on B. Check also that m_B is not right-invariant.

3. (**Finiteness of the Haar measure**).

a) Show that the Haar measure m_G has atoms if and only if G is discrete.

HINT: Assuming that m_G has a point of positive measure show that every other point has the same measure and then study the measure of an arbitrary compact set K with non-empty interior.

b) Show that the Haar measure m_G is finite if and only if G is compact.

HINT: This is essentially Exercise 10.7 with a hint on page 579.

4. (**The modular character**). The aim of this exercise is to "quantify" the difference between left- and right-Haar measures on G.

a) Let $\vartheta : G \to G$ be a continuous automorphism with continuous inverse. Show that there exists a positive number $\mathrm{mod}_G(\vartheta)$ with the property that

$$m_G(\vartheta^{-1}(A)) = \mathrm{mod}_G(\vartheta) m_G(A)$$

for all $A \subset G$ measurable.

b) Applying (a) to the inner automorphisms $\vartheta_g : h \in G \mapsto ghg^{-1}$ for $g \in G$ defines a map $\triangle_G : G \to \mathbb{R}_{>0},\ g \mapsto \mathrm{mod}_G(\vartheta_g)$. Show that for any integrable function f on G and any $h \in G$

$$\int_G f(gh^{-1})\,\mathrm{d}m_G(g) = \triangle_G(h) \int_G f\,\mathrm{d}m_G.$$

Use this to show that $\triangle_G$, known as the *modular character*, is a continuous group homomorphism[1].

HINT: For the latter claim use that for any $f \in C_c(G)$ and any $\varepsilon > 0$ there is a neighborhood of the identity $U \subset G$ with $|f(gh^{-1}) - f(g)| < \varepsilon$ for all $g \in G$ and $h \in U$ (this is uniform continuity of f).

c) We say that G is *unimodular* if $\triangle_G \equiv 1$. Show that G is unimodular under any of the following assumptions:

- G is compact.
- G is abelian.
- G is perfect[2] i.e. G is equal to its commutator subgroup $[G, G]$.

[1] We view $\mathbb{R}_{>0}$ as a topological group where the topology is the standard topology and the group operation is the multiplication.

[2] For example, $\mathrm{SL}_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) : \det(g) = 1\}$ is a perfect group.

5. (**Convolution**). Recall from the lecture that the space of signed measures $\mathcal{M}(G)$ on G is a Banach algebra with a well-behaved involution $\nu \in \mathcal{M}(G) \mapsto \nu^*$ given by $\nu^*(B) = \overline{\nu(B^{-1})}$ for $B \subset G$ measurable. We also view the Banach algebra $L^1_{m_G}(G)$ as a closed subspace of $\mathcal{M}(G)$ where the convolution coincides with the convolution on $\mathcal{M}(G)$.

a) Show that $\mathcal{M}(G)$ is a unital Banach algebra with multiplicative unit the Dirac measure $\delta_{\{e\}}$ at the neutral element $e \in G$.

b) Use Exercise 4 to show that $L^1_{m_G}(G)$ is unital if and only if G is discrete.

Recall from Exercise 5 the notion of a unimodular group. If G is unimodular, we saw in the lecture how to define an involution on $L^1_{m_G}(G)$. We now generalize this to non-unimodular groups.

c) Show the measure $m_G^{(r)}$ on G defined by

$$\int_G f \,\mathrm{d}m_G^{(r)} = \int_G f(g)\triangle(g)^{-1}\,\mathrm{d}m_G(g)$$

is the right Haar measure with $m_G^{(r)}(B) = m_G(B^{-1})$ for all $B \subset G$ measurable.

HINT: Use Exercise 5b) to show that $m_G^{(r)}$ is a right Haar measure. For the second claim check first that right Haar measures are unique up to scalar multiples. Then test against a characteristic function of a small set around the identity given by continuity of the modular character.

d) For $f \in L^1_{m_G}(G)$ we define $f^*(g) = \overline{f(g^{-1})}\triangle(g)^{-1}$ almost everywhere. Show that $f^* \in L^1_{m_G}(G)$ with $\|f^*\|_1 = \|f\|_1$ and with $(f^*)^* = f$. Verify also the following properties:

- Let $f \in L^1_{m_G}(G)$ and let μ_f be the signed measures with $f\,\mathrm{d}m_G = \mathrm{d}\mu_f$. Then $\mathrm{d}\mu_f^* = f^*\,\mathrm{d}m_G$.
- For any $f, g \in L^1_{m_G}(G)$ we have $(f * g)^* = g^* * f^*$.

6. (**Spectrum of a multiplication operator**). Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, $\mathcal{H} = L^2_\mu(X)$ and $g : X \to \mathbb{C}$ be a bounded measurable function. Compute the spectrum of the (bounded) multiplication operator $M_g : \mathcal{H} \to \mathcal{H}$ inside the algebra $B(\mathcal{H})$.

HINT: This is Exercise 11.4 whose statement in the book contains the right answer. There is also a hint on page 581.