D-MATH Functional analysis II Spring 2019
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Exercise sheet 9

1. (**The identity is self-adjoint**). Let $\mathcal{A}$ be a unital C^*-algebra. Show that the identity $\mathbb{1} \in \mathcal{A}$ is self-adjoint and has norm 1.

2. (**Gelfand transform for commutative C^*-algebras**). Read Corollary 11.34 and explain the statements as well as the proof.

3. (**Direct sums**). Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two Banach algebras.

a) Show that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ equipped with componentwise addition and multiplication is a Banach algebra when one considers either of the norms

$$\|(a_1, a_2)\|_1 = \|a_1\| + \|a_2\|, \quad \|(a_1, a_2)\|_\infty = \max\{\|a_1\|, \|a_2\|\}.$$

Also, check that $\mathcal{A}$ is unital (resp. commutative) if and only if both $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital (resp. commutative).

b) Show that for any $a = (a_1, a_2) \in \mathcal{A}$ the spectral radius of a is the maximum of the spectral radii of a_1 and a_2 if $\mathcal{A}$ is unital.

HINT: Use the spectral radius formula.

Assume now that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital and commutative.

c) Show that any character $\chi : \mathcal{A} \to \mathbb{C}$ is of the form $\chi(a_1, a_2) = \chi_1(a_1)$ for a character $\chi_1 : \mathcal{A}_1 \to \mathbb{C}$ or of the form $\chi(a_1, a_2) = \chi_2(a_2)$ for a character $\chi_2 : \mathcal{A}_2 \to \mathbb{C}$.

HINT: Sometimes it is useful to think in terms of maximal ideals. Show that any ideal in the product of two commutative unital rings is a product of ideals in the two rings.

d) Deduce that $\sigma(a) = \sigma(a_1) \cup \sigma(a_2)$ and reprove b).

4. (**Banach algebra of continuous functions**). Let X be a compact metric space and recall that $C(X)$ with pointwise multiplication and the supremum norm is a commutative unital Banach algebra. We aim to show that $\sigma(C(X))$ is homeomorphic to X.

a) Show that for any $x \in X$ the subset $\{f \in C(X) : f(x) = 0\}$ is a maximal ideal.

b) Show that any maximal ideal of $C(X)$ is of this form.

HINT: If $J \subset C(X)$ is not of the form in a), there is for any $x \in X$ a function $f_x \in J$ with $f_x(x) \neq 0$. Now use compactness to construct a function in J that does not vanish anywhere.

c) Use a) and b) to show that any character $C(X) \to \mathbb{C}$ is of the form $\chi_x : f \mapsto f(x)$ for some $x \in X$. Deduce from this the claim of the exercise.

5. (**Stone-Cech compactification**). Recall from the lecture that the Stone-Cech compactification $\beta\mathbb{N}$ is by definition the Gelfand dual $\sigma(\mathcal{A})$ of the commutative unital Banach algebra $\mathcal{A} = \ell^\infty(\mathbb{N})$. By Theorem 11.23, $\beta\mathbb{N}$ is a compact Hausdorff space[1]. We view $\mathbb{N}$ as subspace of $\beta\mathbb{N}$ by associating to $n_0 \in \mathbb{N}$ the character $a = (a_n)_n \mapsto a_{n_0}$.

a) Show that $\mathbb{N}$ is dense in $\beta\mathbb{N}$.

HINT: Verify first that for any $E \subset \mathbb{N}$ and a character $\chi : \mathcal{A} \to \mathbb{C}$ we have $\chi(\mathbb{1}_E) \in \{0, 1\}$. Now given $\varepsilon > 0$ and $a \in \ell^\infty(\mathbb{N})$ show that $\chi(\mathbb{1}_{E_a}) = 0$ where $E_a = \{n \in \mathbb{N} : |a_n - \chi(a)| \geq \varepsilon\}$. For this, consider the sequence b defined by $b_n = 0$ if $n \notin E_a$ and $b_n = \frac{1}{a_n - \chi(a)}$ otherwise.

b) Show that $\ell^\infty(\mathbb{N})$ can be canonically identified with $C(\beta\mathbb{N})$ (where the identification is an identification of Banach algebras).

c) Show that $\beta\mathbb{N}$ is not metrizable.

6. (**Infinite products of Banach algebras**). In this exercise we generalize Exercise 3 to the countable case. For any $n \in \mathbb{N}$ we are given a unital Banach algebra $\mathcal{A}_n$. We define a product of these Banach algebras in two ways and analyse the Gelfand duals.

a) The *direct sum* is the Banach algebra

$$\bigoplus_{n\in\mathbb{N}} \mathcal{A}_n = \Big\{ a = (a_n)_n : a_n \in \mathcal{A}_n \text{ for every } n \text{ and } \|a\|_1 = \sum_{n\in\mathbb{N}} \|a_n\| < \infty \Big\}$$

with componentwise multiplication. Note that it is non-unital. Show that any character $\chi \in \sigma\Big(\bigoplus_{n\in\mathbb{N}} \mathcal{A}_n\Big)$ is of the form $a \mapsto \chi_n(a_n)$ for some $n \in \mathbb{N}$ and some $\chi_n \in \sigma(\mathcal{A}_n)$. Deduce from this that there is a homeomorphism

$$\sigma\Big(\bigoplus_{n\in\mathbb{N}} \mathcal{A}_n\Big) \cong \bigsqcup_{n\in\mathbb{N}} \sigma(\mathcal{A}_n)$$

where the right-hand side is given the disjoint union topology.

HINT: If e_n denotes the n-th vector which is $\mathbb{1}$ at the n-th coordinate and zero otherwise then $\chi(a) = \sum_{n\in\mathbb{N}} \chi(ae_n)$.

[1] Here, the topology is the weak*-topology when viewing $\sigma(\mathcal{A})$ as a closed subset of $\overline{B_1^{\mathcal{A}^*}}$.

b) Assume that the Banach algebras $\mathcal{A}_n$ for $n \in \mathbb{N}$ are unital. The *direct product* is the unital Banach algebra

$$\prod_{n\in\mathbb{N}} \mathcal{A}_n = \Big\{ a = (a_n)_n : a_n \in \mathcal{A}_n \text{ for every } n \text{ and } \|a\|_\infty = \sup_{n\in\mathbb{N}} \|a_n\| < \infty \Big\}$$

where the multiplication is componentwise. We fix for every $n \in \mathbb{N}$ a character $\chi_n \in \sigma(\mathcal{A}_n)$ and obtain an induced a homomorphism of Banach algebras

$$L : \prod_{n\in\mathbb{N}} \mathcal{A}_n \to \ell^\infty(\mathbb{N}), \quad a = (a_n)_n \mapsto (\chi_n(a_n))_n.$$

Given any character $\chi' \in \sigma(\ell^\infty(\mathbb{N})) = \beta\mathbb{N}$ we get a character $\chi' \circ L$ on $\prod_{n\in\mathbb{N}} \mathcal{A}_n$. Show that the so obtained map is continuous and injective.

HINT: There is also a map $\ell^\infty(\mathbb{N}) \to \prod_{n\in\mathbb{N}} \mathcal{A}_n$ where $\alpha = (\alpha_n)_n \in \ell^\infty(\mathbb{N})$ is mapped to $(\alpha_n \mathbb{1})_n \in \prod_{n\in\mathbb{N}} \mathcal{A}_n$.

In some sense a) and b) above show that the Gelfand dual of $\prod_{n\in\mathbb{N}} \mathcal{A}_n$ is much larger than the Gelfand dual of $\bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$. Let us now consider an explicit example. The Banach space $\ell^1(\mathbb{N}_0) \subset \ell^1(\mathbb{Z})$ is turned into a commutative unital Banach algebra when it is equipped with the multiplication coming from multiplication of power series[2]. We then define for $n \in \mathbb{N}$ the unital Banach algebra

$$\mathcal{A}_n = \ell^1(\mathbb{N}_0) / \ell^1(\mathbb{N}_0) e_n$$

where $\ell^1(\mathbb{N}_0) e_n$ is the (closed) principal ideal generated by the n-th coordinate vector e_n. We let $a_n \in \mathcal{A}_n$ be the coset of e_1 and remark that a_n is nilpotent[3].

c) Show that $b = (\frac{1}{n^2} a_n)_n \in \bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$ has spectral radius 0 and check that b is not nilpotent. Prove that $\sigma(\mathcal{A}_n)$ is a one-point set and conclude that the Gelfand dual of $\bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$ is $\mathbb{N}$.

d) Show that $a = (a_n)_n \in \prod_{n\in\mathbb{N}} \mathcal{A}_n$ has spectral radius 1. Use this to show that there are elements of the Gelfand dual of $\prod_{n\in\mathbb{N}} \mathcal{A}_n$ which do not arise as in the construction for part b).

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[2] That is, we define $c = a * b$ for $a, b \in \ell^1(\mathbb{N}_0)$ as the sequence with coefficients of the power series $(\sum_{n=0}^\infty a_n X^n)(\sum_{n=0}^\infty b_n X^n)$ or in other words $c_n = \sum_{k=0}^n a_k b_{n-k}$.

[3] A nilpotent element b of a commutative ring is an element such that $b^n = 0$ for some $n \in \mathbb{N}$. Notice that a matrix over $\mathbb{C}$ is nilpotent if and only if it is conjugate to some upper triangular matrix with zeros on the diagonal. In particular, nilpotent matrices have spectral radius zero.