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Exercise sheet 10

1. (**Non-unital algebra with compact Gelfand dual**). Give an example of a non-unital commutative Banach algebra whose Gelfand dual $\sigma(\mathcal{A})$ is non-empty and compact (and in particular not dense in $\sigma(\mathcal{A}) \cup \{0\}$).

HINT: Consider $\mathcal{A} = \mathbb{C} \times \mathcal{A}'$ where $\mathcal{A}'$ is a nilpotent Banach algebra. That is, for every $a' \in \mathcal{A}'$ there exists $n \in \mathbb{N}$ with $(a')^n = 0$. An example of such a nilpotent algebra can be constructed as in Exercise 6, Sheet 9.

2. (**The Gelfand transform on $\ell^1(\mathbb{Z}^d)$**). Consider the commutative unital Banach algebra $\ell^1(\mathbb{Z}^d)$ endowed with convolution $a * b = (\sum_k a_{n-k} b_k)_k$ and unit $\mathbb{1} = \delta_0$ (see also FA I, Sheet 4).

a) Recall that the Pontryagin dual $\widehat{\mathbb{Z}^d}$ of $\mathbb{Z}^d$ can be identified with $\mathbb{T}^d$ by mapping $t \in \mathbb{T}^d$ to the character $\chi_t : n \in \mathbb{Z}^d \mapsto \mathrm{e}^{2\pi \mathrm{i} n \cdot t}$. Show that under this identification as well as the identification $\sigma(\ell^1(\mathbb{Z}^d)) \simeq \widehat{\mathbb{Z}^d}$ in Proposition 11.38 the Gelfand transform for $\ell^1(\mathbb{Z}^d)$ corresponds to the injective bounded algebra homomorphism

$$\widetilde{\ }: \ell^1(\mathbb{Z}^d) \to C(\mathbb{T}^d), \qquad a = (a_n)_n \mapsto \widetilde{a} = \sum_{n \in \mathbb{Z}^d} a_n \chi_n.$$

REMARK: See also FA I, Sheet 7.

b) Verify that $\tilde{a}(\mathbb{T}^d) = \sigma(a)$ for every $a \in \ell^1(\mathbb{Z}^d)$.

c) Apply b) to show *Wiener's lemma* for $C(\mathbb{T}^d)$: If $f \in C(\mathbb{T}^d)$ has an absolutely convergent Fourier series and has the property that $f(x) \neq 0$ for all $x \in \mathbb{T}^d$ then $\frac{1}{f}$ also has an absolutely convergent Fourier series.

3. (**The Pontryagin dual of $\mathbb{R}^d$**). The aim of this exercise is to show that the Pontryagin dual of $\mathbb{R}^d$ is equal to $\mathbb{R}^d$.

a) We define for any $t \in \mathbb{R}^d$ the map $\chi_t : x \in \mathbb{R}^d \mapsto \mathrm{e}^{2\pi \mathrm{i} x \cdot t}$. Show that χ_t is a character for any t and that $t \in \mathbb{R}^d \mapsto \chi_t \in \widehat{\mathbb{R}^d}$ is injective.

b) Show that any character $\chi \in \widehat{\mathbb{R}^d}$ is of the form χ_t for some $t \in \mathbb{R}^d$.

HINT: For any $s \in \mathbb{R}$ and $\delta > 0$ we have $\int_s^{s+\delta} \chi(x)\,\mathrm{d}x = \chi(s)\int_0^\delta \chi(y)\,\mathrm{d}y$. Use this to show that χ is smooth and find a differential equation that χ satisfies.

c) Show that $t \in \mathbb{R}^d \mapsto \chi_t \in \widehat{\mathbb{R}^d}$ is a homeomorphism.

4. (**An operator with non-empty residual spectrum**). Consider the bounded operator

$$T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad x = (x_n)_n \mapsto (0, x_1, x_2, \ldots).$$

By FAI, Sheet 3 the discrete spectrum $\sigma_{\text{disc}}(T)$ of T (i.e. the set of eigenvalues of T) is empty. Since T is an isometry, $0 \notin \sigma_{\text{appt}}(T)$. However, $0 \in \sigma_{\text{resid}}(T)$ as T is not surjective.

a) Show that the approximate spectrum of T is

$$\sigma_{\text{approx}}(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}.$$

b) Find a formula for the adjoint operator T^* and compute its discrete spectrum.

c) Show that the residual spectrum of T is

$$\sigma_{\text{resid}}(T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}.$$

5. (**Compactness versus discreteness**). Let G be a σ-compact locally compact metrizable abelian group.

a) Show that $\hat{G}$ is discrete if G is compact.

b) Show that $\hat{G}$ is compact if G is discrete.

We note that by Pontryagin duality, the converse results of a),b) above are also true. Let us check one of them by hand:

c) Show that G is discrete if $\hat{G}$ is compact.

HINT: Suppose that G is not discrete. Given any open neighborhood U of $e \in G$ and $g \in U$ non-trivial we may find by separation of characters some $\chi' \in \hat{G}$ such that $\chi'(g) \neq 1$. Take a power of χ' to find a character $\chi \in \hat{G}$ for which $\chi(g)$ is "far away" from 1.

6. (**Closedness of subsets of the spectrum**). Let $\mathcal{H}$ be a separable Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be bounded. By Exercise 5 the residual spectrum is not necessarily closed.

a) Given an example of such an operator T for which the discrete spectrum is not closed.

HINT: Many compact operators have this property. An example is easy to give in a basis of a Hilbert space.

b) Show that the approximate point spectrum $\sigma_{\mathrm{appt}}(T)$ of T is closed.

c) Give an example of an operator T for which the approximate spectrum is not closed.

HINT: Consider $\mathcal{H} = L^2([0,1])^2$ and $T : \mathcal{H} \to \mathcal{H}$ given by $T(f,g)(x) = (xf(x), f(x))$ for $x \in (0,1)$ and $(f,g) \in \mathcal{H}$. Show that $\sigma_{\mathrm{approx}}(T)$ is equal to $(0,1]$.