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Exercise sheet 11

Throughout this exercise sheet, $\mathcal{H}$ is a separable Hilbert space and $\mathcal{A} \subset B(\mathcal{H})$ is a separable commutative unital C^*-subalgebra. For any $v \in \mathcal{H}$ we denote by

$$\langle v\rangle_{\mathcal{A}} = \mathcal{H}_v = \overline{\{av : a \in \mathcal{A}\}}$$

the cyclic subspace generated by v. We denote by μ_v (resp. $\mu_{v,w}$ for $w \in \mathcal{H}$) the diagonal (resp. non-diagonal) spectral measure attached to v (resp. v and w). Note that $\mu_{v,v} = \mu_v$. Recall that there is a unitary isomorphism $\langle v\rangle_{\mathcal{A}} \cong L^2_{\mu_v}(\sigma(\mathcal{A}))$ where multiplication by a° on $L^2_{\mu_v}(\sigma(\mathcal{A}))$ corresponds to applying a on $\langle v\rangle_{\mathcal{A}}$.

1. (**Vanishing of spectral measures**). Let $v, w \in \mathcal{H}$. Show that $\mu_{v,w} = 0$ if and only if the two cyclic subspaces satisfy $\langle v\rangle_{\mathcal{A}} \perp \langle w\rangle_{\mathcal{A}}$.

2. (**Containment of cyclic spaces**). Let $v, w \in \mathcal{H}$.

a) Assume that $w \in \langle v\rangle_{\mathcal{A}}$ and let $f \in L^2_{\mu_v}(\sigma(\mathcal{A}))$ be the function corresponding to w under the isomorphism in the spectral theorem (see above). Show that $\mathrm{d}\mu_w = |f|^2\,\mathrm{d}\mu_v$.

b) Prove that $\mu_w \ll \mu_v$ if and only if there is an isometric embedding $\phi : \langle w\rangle_{\mathcal{A}} \to \langle v\rangle_{\mathcal{A}}$ such that $\phi(ax) = a\phi(x)$ for all $a \in \mathcal{A}$ and $x \in \langle w\rangle_{\mathcal{A}}$.

HINT: Assuming that $\mu_w \ll \mu_v$ let f be the square-root of the Radon-Nikodym derivative. Check that $f \in L^2_{\mu_v}(\sigma(\mathcal{A}))$ and use the spectral theorem as above for this f.

3. (**Spectral theorem for normal operators**). Recall that in the lecture we generalized Theorem 12.33 to Theorem 12.60 which we claimed to be more general. Amongst other things, the aim of this exercise is to show that this is true.

a) Let $T \in B(\mathcal{H})$ be a normal operator and consider the algebra

$$\mathcal{A}_T = \overline{\langle T^m (T^*)^n : m, n \in \mathbb{N}_0\rangle}.$$

Show that the map $\chi \in \sigma(\mathcal{A}_T) \mapsto \chi(T) \in \sigma_{\mathcal{A}_T}(T)$ is a homeomorphism, where $\sigma_{\mathcal{A}_T}(T)$ denotes the spectrum of T as an element of the algebra $\mathcal{A}_T$.

b) Show that $\sigma_{\mathcal{A}_T}(T) = \sigma_{B(\mathcal{H})}(T)$. We can thus simply write $\sigma(T)$ for the spectrum.

HINT: The spectral theorem realizes the operator T as a multiplication operator. For the latter, we know the spectrum from Exercise 6 in Sheet 8.

c) Combine a) with Theorem 12.60 to prove Theorem 12.33. Check also that for any $v \in \mathcal{H}$ there is a uniquely defined μ'_v on $\sigma(T)$ such that

$$\langle T^m (T^*)^n v, v\rangle = \int_{\sigma(T)} z^m \overline{z}^n \, \mathrm{d}\mu'_v(z).$$

d) Study the measurable functional calculus for the algebra $\mathcal{A}_T$ and comment on how to prove Theorem 12.34.

4. (**An example**). Consider the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$ and the bounded operator

$$T : x \in \ell^2(\mathbb{Z}) \mapsto Tx, \quad (Tx)_n = \tfrac{1}{2}(x_{n-1} + x_{n+1}) \text{ for all } n \in \mathbb{Z}.$$

a) Show that T is self-adjoint (in particular normal) and that the spectrum $\sigma(T)$ of T is equal to $[-1, 1]$.

HINT: Start by computing the spectrum of the right-shift operator $S_r : \ell^2(\mathbb{Z}) \mapsto \ell^2(\mathbb{Z})$ and its adjoint operator $S_r^* = S_\ell$ which is the left-shift operator. Then check that the spectrum of T satisfies $\sigma(T) = \{\frac{\lambda+\overline{\lambda}}{2} : \lambda \in \sigma(S_r)\}$.

b) Find an explicit finite measure space (X, μ) such that $\mathcal{H} \cong L^2(X, \mu)$ as in Theorem 12.33.

HINT: The space X consists of two copies of the spectrum $\sigma(T)$ of T, the two copies arising from considering the symmetry $(x_n)_n \mapsto (x_{-n})_n$. You may use the identity

$$\int_{-1}^{1} \cos(\pi(n-2m)x) \sin(\pi x)^n \, \mathrm{d}x = 2^{-n+1}\binom{n}{m}.$$

for $m \leq n$ and even n.

5. (**Maximal spectral type**). The aim of this exercise is to construct a spectral measure μ_v for $v \in \mathcal{H}$ satisfying $\mu_w \ll \mu_v$ for any $w \in \mathcal{H}$. The measure class of μ_v is called the *maximal spectral type* of $\mathcal{A}$.

a) Let $v \in \mathcal{H}$ and let $w \in \langle v\rangle_{\mathcal{A}}^{\perp}$. Show that $\mu_{v+w} = \mu_v + \mu_w$.

b) Show that there exists $v \in \mathcal{H}$ with the property that $\mu_w \ll \mu_v$ for any $w \in \mathcal{H}$.

This concludes our construction of the maximal spectral type. Note that the cyclic subspace of any vector whose measure class is the maximal spectral type contains all the information by Exercise 2.

6. (**Spectral theorem for compact, normal operators**). Let $T : \mathcal{H} \to \mathcal{H}$ be a compact normal operator.

a) Let μ be a finite measure on $\mathbb{C}$. When is the operator

$$M_I : L^2_\mu(\mathbb{C}) \to L^2_\mu(\mathbb{C}),\ f \mapsto (z \mapsto zf(z))$$

compact?

b) State and prove a spectral theorem for T similar to Theorem 6.27.