D-MATH Functional analysis II Spring 2019
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Exercise sheet 12

Throughout this exercise sheet, G is a locally compact σ-compact metrizable abelian group and π is a unitary representation of G on a separable Hilbert space $\mathcal{H}$. Recall $L^1(G)$ is a commutative Banach algebra and that we defined for $f \in L^1(G)$ an operator $\pi_*(f)$ on $\mathcal{H}$ uniquely determined by the property

$$\langle \pi_*(f)v, w\rangle = \int_G f(g)\, \langle \pi_g v, w\rangle \, \mathrm{d}m(g)$$

for all $v, w \in \mathcal{H}$. Here, m is a Haar measure on G.

1. (**Properties of the L^1-convolution operator**). Show that

$$\pi_* : f \in L^1(G) \mapsto \pi_*(f) \in B(\mathcal{H})$$

is a homomorphism of Banach algebras.

2. (**Multiplication operator on the Pontryagin dual**). Let μ be a σ-finite measure on the Pontryagin dual $\hat{G}$. For $g \in G$ and $f \in L^2_\mu(\hat{G})$ we define $M_g f \in L^2_\mu(\hat{G})$ by

$$M_g f(\chi) = \chi(g) f(\chi).$$

Show that M_g is a unitary representation of G on $L^2_\mu(\hat{G})$.

3. (**Roots of an operator**). Let $\mathcal{H}$ be a separable Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator. Suppose that T is a positive operator i.e. $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and let $n \in \mathbb{N}$.

a) Show that $\sigma(T) \subset [0, \infty)$.

b) Apply the functional calculus to find a self-adjoint and positive bounded operator $S : \mathcal{H} \to \mathcal{H}$ with the property $S^n = T$.

REMARK: Actually, the requirement that $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$ implies that T is self-adjoint.

We now wish to show uniqueness of the operator S as above.

c) We begin with a “special” case. Suppose that $\mathcal{H} = L^2_\mu(X)$ where (X, μ) is a finite measure space and that M_g, M_h are two positive multiplication operators for $g, h \in L^\infty_\mu(X)$ with $g^n = h^n$. Show that $g = h$.

d) Now let S be as in b) and let $S' : \mathcal{H} \to \mathcal{H}$ be self-adjoint and positive with $(S')^n = T$. Show that $S = S'$.

HINT: Apply (FC5) to show that S and S' commute and then consider the unital algebra $\mathcal{A}$ generated by S and S'.

4. (**Schur's lemma**). Let G be a topological group, let $\mathcal{H}_1, \mathcal{H}_2$ be two separable Hilbert space and let π_1, π_2 be unitary representations of G on $\mathcal{H}_1$ respectively $\mathcal{H}_2$. We let $B : \mathcal{H}_1 \to \mathcal{H}_2$ be an operator with the property that $B\pi_1(g) = \pi_2(g)B$ for all $g \in G$ (one says in this case that B *intertwines* π_1 and π_2).

a) Suppose that π_1 is irreducible i.e. that there are no invariant closed subspaces of $\mathcal{H}_1$ other than $\{0\}$ and $\mathcal{H}_1$. Show that there is some $\lambda \geq 0$ such that $B^*B = \lambda I_{\mathcal{H}_1}$.

b) Suppose that $\mathcal{H}_1 = \mathcal{H}_2$ and $\pi_1 = \pi_2$. Prove that $B = \lambda I_{\mathcal{H}_1}$ for some $\lambda \in \mathbb{C}$.

HINT: This is Exercise 12.58 in the book.

5. (**Spectral measures for** $\mathbb{R}^d$). Let π be a unitary representation of $\mathbb{R}^d$ on a Hilbert space $\mathcal{H}$ and let $v \in \mathcal{H}$. Show that there is a unique finite measure μ_v on $\widehat{\mathbb{R}^d}$ such that

$$\langle \pi_x v, v \rangle = \int_{\widehat{\mathbb{R}^d}} \chi(x) \,\mathrm{d}\mu_v(\chi)$$

for all $x \in \mathbb{R}^d$.

HINT: Revisit the proof of Corollary 12.81.

NOTE: As formulated, the statement also applies other groups. Since we may identify $\mathbb{R}^d \cong \widehat{\mathbb{R}^d}$ via $t \in \mathbb{R}^d \mapsto \chi_t$ where $\chi_t(x) = \mathrm{e}^{2\pi \mathrm{i}\langle t,x\rangle}$ the exercise shows that there is a unique finite measure μ'_v on $\mathbb{R}^d$ such that

$$\langle \pi_x v, v \rangle = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i}\langle t,x\rangle} \,\mathrm{d}\mu'_v(t).$$

6. (**Commutant of the left-regular representation on** $\mathbb{R}$). We consider the left-regular representation π of $\mathbb{R}$ on $L^2(\mathbb{R})$ where we use the Lebesgue measure implicitly. Recall that it is given by $\pi_y f(y) = f(y - x)$ for any $x, y \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$. Also, we may map $L^1(\mathbb{R}) \oplus \mathbb{C}$ into a commutative unital C^*-subalgebra $\mathcal{A}$ of $B(L^2(\mathbb{R}))$ via π_*.

a) Show that $L^2(\mathbb{R})$ is cyclic.

HINT: There are two different ways in which one can be cyclic (i.e. cyclic for $\mathcal{A}$ or cyclic for the unitary representation) and they agree here. To prove the statement, consider the characteristic function of an interval.

Suppose now that $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded operator with the $\pi_x \circ T = T \circ \pi_x$ for all $x \in \mathbb{R}$.

b) Show that there is $\psi \in \mathscr{L}^\infty(\mathbb{R})$ with the property that $T = \mathrm{FC}(\psi)$.

HINT: Consider first the following problem. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be bounded with the property that $T \circ M_\phi = M_\psi \circ T$ for all $\phi \in \mathscr{L}^\infty(\mathbb{R})$. Then $T = M_\psi$ for some $\psi \in \mathscr{L}^\infty(\mathbb{R})$.