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Solutions for exercise sheet 1

1. **a)** Notice first that the map is a priori not even well-defined. As this is the first exercise, we will be very careful with the formal reasonings.

We begin by remarking that the restriction operator

$$R_V : L^2(U) \mapsto L^2(V)$$

is clearly well-defined and bounded and in fact, $\|f|_V\|_{L^2} \leq \|f\|_{L^2}$ for any $f \in L^2(U)$. This induces a bounded restriction operator

$$R'_V : \bigoplus_{\|\alpha\|_1 \leq k} L^2(U) \to \bigoplus_{\|\alpha\|_1 \leq k} L^2(V)$$

by applying R_V componentwise. Recall that $H^k(U)$ is defined as the closure of the subspace

$$W_U = \{(\partial_\alpha f)_{\|\alpha\|_1 \leq k} : f \in C^\infty(U) \text{ and } \partial_\alpha f \text{ for all } \|\alpha\|_1 \leq k\}$$

and analogously for $H^k(V)$ and W_V. We claim that the image under R'_V of W_U lies inside W_V. This follows directly as $f|_V \in C^\infty(V)$ and

$$\partial_\alpha(f|_V) = (\partial_\alpha f)|_V = R_V(\partial_\alpha f)$$

for any $f \in W_U$. From the claim it follows that we have a bounded restriction operator $R'_V : W_U \to H^k(V)$ which extends uniquely to a bounded operator $\mathrm{res}_V : H^k(U) \to H^k(V)$.

It remains to explain why we should call this a restriction operator. Recall that we may identify $(f_\alpha)_{\|\alpha\|_1 \leq k} \in H^k(U)$ with the representing function f_0 (see Lemma 5.10) – we will in fact often do this implicitly. The other components f_α are then simply the weak derivatives which exist by assumption. Note that if $f \in W_U$ then for any $\phi \in C_c^\infty(V)$ by partial integration

$$\begin{aligned}\int_V \mathrm{res}_V(f)_0 \partial_\alpha \phi \, \mathrm{d}x = \int_V f \partial_\alpha \phi \, \mathrm{d}x &= (-1)^{\|\alpha\|_1} \int_V (\partial_\alpha f) \phi \, \mathrm{d}x \\ &= (-1)^{\|\alpha\|_1} \int_V \mathrm{res}_V(f)_\alpha \phi \, \mathrm{d}x.,\end{aligned}$$

This equation extends to the completion and hence the weak derivatives of $f|_V$ are the restriction of the weak derivatives of f to V. This proves the remaining claim.

b) Having explained the main machinery in a) we will be brief. Recall that $H_0^k(V)$ is the closure of $C_c^\infty(U)$ viewed as a subspace of $H^k(U)$ by $f \mapsto (\partial_\alpha f)$. If $f \mapsto \tilde{f}$ denotes the extension by zero, this induces a bounded operator $\bigoplus_{\|\alpha\|_1 \leq k} L^2(V) \to \bigoplus_{\|\alpha\|_1 \leq k} L^2(U)$ and the image of $C_c^\infty(V)$ lies in $C_c^\infty(U)$ (under the identification already made). This gives an extension operator $H_0^k(V) \to H_0^k(U)$ by extension to the completion. A quick calculation shows that the extension of $\partial_\alpha f$ is equal to ∂_α of the extension, which then extends to the completion and shows that $H_0^k(V) \to H_0^k(U)$ is indeed the extension by zero.

2. This is an exercise on Sheet 13 of FAI, we repeat here the solution for convenience of the reader.

Let us first note that the formula

$$\int_{\mathbb{T}^d} \psi(x)\partial_\alpha f(x)\,\mathrm{d}x = (-1)^{\|\alpha\|_1} \int_{\mathbb{T}^d} \partial_\alpha \psi(x) f(x)\,\mathrm{d}x \tag{1}$$

holds by partial integration for any $f \in C^\infty(\mathbb{T}^d)$ and $\psi \in C^\infty(\mathbb{T}^d)$.

- Assume that $f \in H^k(\mathbb{T}^d)$ and denote by $(f_\alpha)_{\|\alpha\|_1 \leq k}$ the corresponding tuple of functions in $L^2(\mathbb{T}^d)$. By definition, there exists a sequence of functions $g_k \in C^\infty(\mathbb{T}^d)$ so that $\partial_\alpha g_k \to f_\alpha$ in L^2 as $k \to \infty$ for any α. Now note that the left-hand side of (1) when applied to g_k must converge to

$$\int_{\mathbb{T}^d} \psi(x) f_\alpha(x)\,\mathrm{d}x.$$

  This is in essence a consequence of the Cauchy-Schwarz inequality. Applying the same reasoning to the right-hand side we obtain

$$\int_{\mathbb{T}^d} \psi(x) f_\alpha(x)\,\mathrm{d}x = (-1)^{\|\alpha\|_1} \int_{\mathbb{T}^d} \partial_\alpha \psi(x) f(x)\,\mathrm{d}x$$

  for any $\psi \in C^\infty(\mathbb{T}^d)$. Since $f_\alpha \in L^2(\mathbb{T}^d)$ we have thus found the α-weak derivative of f.

- Now suppose that all α-weak derivatives for $\|\alpha\|_1 \leq k$ exist and denote them by f_α. Let us compute the Fourier series of f_α: Applying the definition of the weak derivative to $\psi = \chi_{-n}$ for $n \in \mathbb{Z}^d$ we obtain that

$$a_n(f_\alpha) = \int_{\mathbb{T}^d} \chi_{-n}(x) f_\alpha(x)\,\mathrm{d}x = (-1)^{\|\alpha\|_1} \int_{\mathbb{T}^d} \partial_\alpha \chi_{-n}(x) f(x)\,\mathrm{d}x.$$

  Now note that

$$\begin{aligned} \partial_\alpha \chi_{-n}(x) &= (2\pi \mathrm{i})^{\|\alpha\|_1} (-n_1)^{\alpha_1} \cdots (-n_d)^{\alpha_d} \chi_{-n}(x) \\ &= (2\pi \mathrm{i})^{\|\alpha\|_1} (-1)^{\|\alpha\|_1} n^\alpha \chi_{-n}(x) \end{aligned}$$

and so

$$a_n(f_\alpha) = (2\pi \mathrm{i})^{\|\alpha\|_1} n^\alpha a_n(f).$$

Let $g_N = \sum_{n=-N}^{N} a_n(f)\chi_n$. Since $f_\alpha \in L^2(\mathbb{T}^d)$ and the Fourier series of f_α has the above shape, $\partial_\alpha g_N \to f_\alpha$ in L^2 for any α with $\|\alpha\|_1 \le k$. This shows that $f \in H^k(\mathbb{T}^d)$ as desired.

3. We follow the hint but begin by remarking that more sofisticated tools (such as the convolution) allow for an alternative exposition. Ours here will be mostly elementary.

Claim 1: If f is a continuous function on $[0,1)$ and is continuously differentiable on $(0,1)$ with $f' \in L^2((0,1))$ then $f \in H^1((0,1))$.

To prove this, let $\varepsilon > 0$ and let $\psi \in C_c^\infty((0,1))$ be such that $\|f' - \psi\|_2 < \varepsilon$ (cf. Exercise 5). We now choose $\Psi \in C^\infty((0,1))$ with $\Psi' = \psi$ and $\Psi(\frac{1}{2}) = f(\frac{1}{2})$. We apply (5.7) to the function $f - \Psi$ and obtain for $x \in [0, \frac{1}{2})$

$$|f(x) - \Psi(x)| = |(f-\Psi)(\tfrac{1}{2}) - (f-\Psi)(x)| \le \|(f-\Psi)'\|_{L^2}$$

and similarly for all other points x which proves that $\|f - \Psi\|_{L^2} < \varepsilon$ by integrating the square of the above inequality. This concludes the proof of the claim.

We now turn to the problem at hand. Let us call a function $f : U \to \mathbb{R}$ *radial* if it is of the form $f(x) = f^{\mathrm{rad}}(\|x\|)$ where $f^{\mathrm{rad}} : [0,1) \to \mathbb{R}$ is a function of the radius.

Claim 2: If $f \in C^\infty(U) \cap H^1(U)$ is radial then

$$\int_0^1 |(f^{\mathrm{rad}})'(r)|^2 r^{d-1} \,\mathrm{d}r < \infty.$$

For this claim, notice first that for $x \neq 0$

$$\partial_{e_i} f(x) = 2x_i \tfrac{1}{2\|x\|} (f^{\mathrm{rad}})'(\|x\|) = \tfrac{x_i}{\|x\|} (f^{\mathrm{rad}})'(\|x\|)$$

by the chain rule. By assumption $\partial_{e_i} f \in L^2(U)$ for every i and thus

$$\|\partial_{e_1} f\|_{L^2}^2 + \ldots + \|\partial_{e_d} f\|_{L^2}^2 < \infty.$$

We now compute this sum:

$$\begin{aligned}\|\partial_{e_1} f\|_{L^2}^2 + \ldots + \|\partial_{e_d} f\|_{L^2}^2 &= \int_U \left(\tfrac{x_1^2}{\|x\|^2} + \ldots + \tfrac{x_d^2}{\|x\|^2}\right)(f^{\mathrm{rad}})'(\|x\|) \,\mathrm{d}x \\ &= \int_U |(f^{\mathrm{rad}})'(\|x\|)|^2 \,\mathrm{d}x = \mathrm{vol}(\mathbb{S}^{d-1}) \int_0^1 |(f^{\mathrm{rad}})'(r)|^2 r^{d-1} \,\mathrm{d}r\end{aligned}$$

by using spherical coordinates. This proves this claim.

If $f_\alpha \in H^1(U)$ then applying the definition of the weak derivative to f_α and smooth functions with support in $U \setminus \{0\}$ shows that

$$\boldsymbol{\partial}_{e_i} f_\alpha \underset{a.e.}{=} \partial_{e_i} f_\alpha$$

The argument for Claim 2 applied to annuli filling out more and more of U shows that

$$\int_0^1 (f_\alpha^{\mathrm{rad}})'(r) r^{d-1} \,\mathrm{d}r \ll \|\partial_{e_1} f\|_{L^2}^2 + \ldots + \|\partial_{e_d} f\|_{L^2}^2$$

where the left hand side is

$$\int_0^1 (f_\alpha^{\mathrm{rad}})'(r)^2 r^{d-1} \,\mathrm{d}r = \int_0^1 ((r^\alpha)')^2 r^{d-1} \,\mathrm{d}r = \alpha \int_0^1 r^{2\alpha-2+d-1} \,\mathrm{d}r$$

which exists if and only if $2\alpha - 2 + d - 1 > -1$ or in other words $\alpha > \frac{2-d}{2}$. We have thus found a necessary condition.

To show that it is sufficient, we will construct a smooth radial function approximating f and its derivatives. To do this, we apply the proof of Claim 1 under a slight adaption. To simplify notation, we set $f = f_\alpha^{\mathrm{rad}} : r \mapsto r^\alpha$. By assumption, we know that

$$\int_0^1 f'(r)^2 r^{d-1} \,\mathrm{d}r < \infty \text{ and } \int_0^1 f(r)^2 r^{d-1} \,\mathrm{d}r < \infty$$

The density r^{d-1} does not allow us to use Claim 1 directly. Instead, we will use a *cut-off* to apply the argument of Claim 1. Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\int_0^\delta f'(r)^2 r^{d-1} \,\mathrm{d}r < \varepsilon^2 \text{ and } \sup_{r\in[0,\delta]} |f(r)|^2 r^{d-1} < \varepsilon^2.$$

We know (by continuity of f and f' on $[\delta, 1]$) that $f, f' \in L^2((\delta, 1))$. In particular, we may pick $\psi_0 \in C_c^\infty((\delta, 1))$ such that $\|f' - \psi_0\|_{L^2} < \varepsilon$. We extend ψ_0 by zero to obtain a function in $C_c^\infty((0, 1))$ that we still denote by ψ_0. Now pick $\Psi \in C^\infty([0, 1])$ such that $\Psi' = \psi_0$ and $\Psi(\delta) = f(\delta)$. Then

$$\begin{aligned}\int_0^\delta |f(r) - \Psi(r)|^2 r^{d-1} \,\mathrm{d}r &\le \int_0^\delta 2(|f(r)|^2 + |\Psi(r)|^2) r^{d-1} \,\mathrm{d}r \\ &= 2\int_0^\delta (|f(r)|^2 + |f(\delta)|^2) r^{d-1} \,\mathrm{d}r < 2\delta\varepsilon^2.\end{aligned}$$

For $r \in [\delta, 1]$ we estimate

$$\begin{aligned}|f(r) - \Psi(r)| = |f(r) - \Psi(r) - (f(\delta) - \Psi(\delta))| &\le \int_0^r |f'(t) - \psi_0(t)| \,\mathrm{d}t \\ &\ll \|f' - \psi_0\|_{L^2} < \varepsilon\end{aligned}$$

to obtain

$$\int_\delta^1 |f(r) - \Psi(r)|^2 r^{d-1}\,\mathrm{d}r \ll \varepsilon^2.$$

Overall, we have constructed $\Psi \in C^\infty([0,1])$ with

$$\int_0^1 |f(r) - \Psi(r)|^2 r^{d-1}\,\mathrm{d}r \ll \varepsilon^2 \text{ and } \int_0^1 |f'(r) - \Psi'(r)|^2 r^{d-1}\,\mathrm{d}r \ll \varepsilon^2$$

where $f = f_\alpha^{\mathrm{rad}}$.

We now let $\phi \in C^\infty(U) \cap H^1(U)$ be the radial function with $\phi^{\mathrm{rad}} = \Psi$. The same argument as in the proof of Claim 2 shows that

$$\|\boldsymbol{\partial}_{e_1}(f_\alpha - \phi)\|_{L^2}^2 + \ldots + \|\boldsymbol{\partial}_{e_d}(f_\alpha - \phi)\|_{L^2}^2 \ll \int_0^1 |\Psi'(r) - (f_\alpha^{\mathrm{rad}})'(r)|^2 r^{d-1}\,\mathrm{d}r \ll \varepsilon^2.$$

Also, by the same spherical substitution

$$\|f_\alpha - \phi\|_{L^2}^2 \ll \int_0^1 |\Psi(r) - (f_\alpha^{\mathrm{rad}})(r)|^2 r^{d-1}\,\mathrm{d}r \ll \varepsilon^2$$

which concludes the proof.

4. **a)** A product of smooth functions is smooth, so one direction is clear. Assume that $\psi_p f$ is smooth for every $p \in U$. Then

$$(\psi_p f)|_{B_r(p)} = \psi_p|_{B_r(p)} f|_{B_r(p)} = f|_{B_r(p)}$$

is smooth for every $p \in U$, which implies that f is smooth. (Smoothness is a local condition).

b) By the product rule, we have that $\partial_{e_j}^\ell(\psi_p f)$ exists and is equal to

$$\partial_{e_j}^\ell(\psi_p f) = \sum_{i=0}^{\ell} \binom{\ell}{i} (\partial_{e_j}^i \psi_p)(\partial_{e_j}^{\ell-i} f).$$

The right hand side is certainly continuous.

c) Note that $U' = (-R/2, R/2)^d$ can be viewed as an open subset of $\mathbb{R}^d/R\mathbb{Z}^d$ as the projection map $\pi : \mathbb{R}^d \to \mathbb{R}^d/R\mathbb{Z}^d$ restricted to U' is injective and $\pi^{-1}(\pi(U')) = \bigsqcup_{n\in\mathbb{Z}^d} n + U'$ is open. Since f is compactly supported in U' by the previous reductions, the function $\tilde{f}$ is continuous and are all of its partial derivatives in the coordinate directions. We are thus in the setting of Exercise 6c) on Sheet 6 (FAI) which implies that $\tilde{f}$ is smooth. But then so is $\tilde{f}|_{U'} = f$.

5. **a)** This is an exercise in first year analysis. The main input is that the limit

$$\lim_{t\nearrow 0} \frac{1}{p(t)} e^{\frac{1}{t}}$$

exists for any polynomial p and is zero.

b) The function j and j_ε are smooth as compositions of smooth functions. Assume that $j_\varepsilon(x) \neq 0$ and hence $j(\frac{x}{\varepsilon}) \neq 0$. Note that since $\psi(t) = 0$ for $t > 0$ we have $j(y) = 0$ for y with $\|y\|_2^2 - 1 > 0$ i.e. $\|y\| > 1$. Thus, $j(\varepsilon x) \neq 0$ implies that $\|\frac{x}{\varepsilon}\| \leq 1$ which implies the claim about the support.

c) We note first that the convolution is well-defined. Indeed, for any $x \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} f(y) j_\varepsilon(x-y)\,\mathrm{d}y = \int_{\overline{B_\varepsilon(x)}} f(y) j_\varepsilon(x-y)\,\mathrm{d}y$$

using b). To show smoothness, notice that if $B_\delta(x)$ is a ball around $x \in \mathbb{R}^d$ then for any $x' \in B_\delta(x)$

$$f_\varepsilon(x') = \int_{\overline{B_\varepsilon(x')}} f(y) j_\varepsilon(x'-y)\,\mathrm{d}y = \int_{\overline{B_{\varepsilon+\delta}(x)}} f(y) j_\varepsilon(x'-y)\,\mathrm{d}y.$$

We have thus reinterpreted f_ε as a parameter integral over a compact region. These are smooth functions by classical analysis.

To show uniform convergence on compacta, let $K \subset \mathbb{R}^d$ be compact and let $\eta > 0$. For convenience let $K' = K + \overline{B_1(0)}$ be a slightly blown-up version of K By uniform continuity of $f|_{K'}$ we can choose $\varepsilon \in (0,1)$ such that for all $x, y \in K'$

$$\|x-y\| \leq \varepsilon \implies |f(x) - f(y)| < \eta.$$

Notice that if $x \in K$ and $\|y - x\| \leq \varepsilon$ then $y \in K'$ and so the above implication holds. We now estimate for $x \in K$

$$\begin{aligned}
|f(x) - f_\varepsilon(x)| &= \left| \int_{\mathbb{R}^d} f(y) j_\varepsilon(x-y)\,\mathrm{d}y - \int_{\mathbb{R}^d} f(x) j_\varepsilon(x-y)\,\mathrm{d}y \right| \\
&\leq \int_{\mathbb{R}^d} |f(y) - f(x)| j_\varepsilon(x-y)\,\mathrm{d}y \\
&= \int_{\overline{B_\varepsilon(x)}} |f(y) - f(x)| j_\varepsilon(x-y)\,\mathrm{d}y \\
&< \eta \int_{\overline{B_\varepsilon(x)}} j_\varepsilon(x-y)\,\mathrm{d}y = \eta
\end{aligned}$$

as desired where we used that b) again.

For the remaining claim, note that if $x \notin \text{supp}(f) + \overline{B_\varepsilon(0)}$ then for any $y \in \overline{B_\varepsilon(x)}$ we have $y \notin \text{supp}(f)$. This shows that for such an x

$$f_\varepsilon(x) = \int_{\mathbb{R}^d} f(y) j_\varepsilon(x-y)\,\mathrm{d}y = \int_{\overline{B_\varepsilon(x)}} f(y) j_\varepsilon(x-y)\,\mathrm{d}y = 0$$

as claimed.

d) It suffices to show that for $f \in C_c(U)$ and for $\eta > 0$ there is $\tilde{f} \in C_c^\infty(U)$ with $\|f - \tilde{f}\|_{L^p} < \eta$. Let $K = \text{supp}(f)$ and $K' = K + \overline{B_\delta(0)}$ where $\delta > 0$ is small enough such that $K' \subset U$. Define f_ε as in c). We know from c) that $\text{supp}(f_\varepsilon) \subset K'$ if $\varepsilon < \delta$ and that $f_\varepsilon \to f$ uniformly on compact sets and in particular uniformly on K'. Let ε be small such that

$$\|f|_{K'} - f_\varepsilon|_{K'}\|_\infty < \eta.$$

Then

$$\|f - f_\varepsilon\|_{L^p}^p = \int_U |f(x) - f_\varepsilon(x)|^p\,\mathrm{d}x = \int_{K'} |f(x) - f_\varepsilon(x)|^p\,\mathrm{d}x \leq \text{vol}(K')\eta^p.$$

This proves the claim.

6. We first adapt 5c) to obtain the following claim.

Claim 1: Let $g \in L^2(U)$ and extend it by zero to a function on $\mathbb{R}^d$ (or equivalence class of such). Then $g * j_\varepsilon$ is smooth and $\|g * j_\varepsilon\|_{L^2} \leq \|g\|_{L^2}$. Furthermore, the restriction of $g * j_\varepsilon$ to U converges in $L^2(U)$ to g.

We leave the first part of the claim to the reader; it is a standard statement from measure theory. If $\delta > 0$ and $f \in C_c(U)$ is such that $\|f - g\|_{L^2} < \delta$ then

$$\begin{aligned}\|g * j_\varepsilon - g\|_{L^2(U)} &\leq \|g * j_\varepsilon - f * j_\varepsilon\|_{L^2(U)} + \|f * j_\varepsilon - f\|_{L^2(U)} + \|f - g\|_{L^2(U)} \\ &\leq 2\delta + \|f * j_\varepsilon - f\|_{L^2(U)}.\end{aligned}$$

We may thus assume that $g \in C_c(U)$. For $\varepsilon > 0$ small enough, $g * j_\varepsilon \in C_c(U)$ by Exercise 5c). The claim now follows directly from the uniform convergence on compacta statement in 5c).

Instead, let us prove the following claim regarding weak derivatives.

Claim 2: If $g \in L^2(U)$ has a weak partial derivative $g_i \in L^2(U)$ in direction e_i then so does $g * j_\varepsilon$ restricted to U_ε where U_ε is the set of points in U with distance $> \varepsilon$ to ∂U. Moreover we have

$$\boldsymbol{\partial}_{e_i}(g * j_\varepsilon) = (g_i) * j_\varepsilon$$

on U_ε.

To prove this, note that the right hand side is in L^2 by Claim 1. We thus only need to show that $(g_i) * j_\varepsilon$ satisfies the definition of the weak derivative for $g * j_\varepsilon$. Let $\psi \in C_c^\infty(U_\varepsilon)$. We then compute using properties of the convolution

$$\begin{aligned}\int_{U_\varepsilon} g_i * j_\varepsilon(x)\psi(x)\,\mathrm{d}x &= \int_{U_\varepsilon} \psi(x) \int_{\mathbb{R}^d} g_i(y) j_\varepsilon(y-x)\,\mathrm{d}y\ \mathrm{d}x \\ &= \int_{\mathbb{R}^d} g_i(y) \int_{U_\varepsilon} \psi(x) j_\varepsilon(y-x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} g_i(y) \psi * j_\varepsilon(y)\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^d} g(y)\partial_{e_i}(\psi * j_\varepsilon)(y)\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^d} g(y)(\partial_{e_i}\psi) * j_\varepsilon(y)\,\mathrm{d}y \\ &= \ldots = -\int_{U_\varepsilon} g * j_\varepsilon(x)(\partial_{e_i}\psi)\,\mathrm{d}x.\end{aligned}$$

Notice that we used that $\psi \in C_c^\infty(U_\varepsilon)$ as this implies that $\psi * j_\varepsilon \in C_c^\infty(U)$ which was needed above when we applied the definition of the weak derivative. This concludes the proof of the claim.

We now turn to the actual exercise. Let f be as in the exercise and let $\lambda \in (0,1)$. Define

$$f^\lambda : \tfrac{1}{\lambda}U \ni x \mapsto f(\lambda x).$$

Notice that by assumption on U, $\frac{1}{\lambda}U \supset \overline{U}$. If we choose $\varepsilon > 0$ smaller than the distance of $\overline{U}$ to the boundary of $\frac{1}{\lambda}U$, the smooth function $f^\lambda * j_\varepsilon$ and its derivatives are in $L^2(U)$ by Claim 2. It remains to show that for any $\delta > 0$ we may choose λ and ε so that $f^\lambda * j_\varepsilon$ is δ-close in L^2 to f and similarly for its weak derivatives. Notice that the weak derivative in the i-th direction of f^λ is given by $\lambda^d f_i^\lambda$.

The main missing ingredient is the following:

Claim 3: For any $g \in L^2(U)$ the restriction of g^λ to U converges to g in $L^2(U)$ as $\lambda \to 1$.

For continuous, compactly supported functions on U this is rather directly proven. Otherwise, one can apply the density of $C_c(U)$ in $L^2(U)$.

Let $\lambda \in (0,1)$ with $\|f^\lambda - f\|_{L^2(U)} < \delta$ and $\|f_i^\lambda - f_i\|_{L^2(U)} < \delta$ for $i = 1, \ldots, d$. Furthermore, let $\varepsilon > 0$ be smaller than the distance of $\overline{U}$ to the boundary of $\frac{1}{\lambda}U$ with the property that $\|f * j_\varepsilon - f\|_{L^2(U)} < \delta$ and $\|f_i * j_\varepsilon - f_i\|_{L^2(U)} < \delta$ for $i = 1, \ldots, d$.

Then

$$\begin{aligned}\|f^\lambda * j_\varepsilon - f\|_{L^2(U)} &\le \|f^\lambda * j_\varepsilon - f * j_\varepsilon\|_{L^2(U)} + \|f * j_\varepsilon - f\|_{L^2(U)} \\ &= \|(f^\lambda - f) * j_\varepsilon\|_{L^2(U)} + \|f * j_\varepsilon - f\|_{L^2(U)} \\ &< \|f^\lambda - f\|_{L^2(U)} + \delta < 2\delta\end{aligned}$$

and similarly

$$\begin{aligned}\|\lambda^d f_i^\lambda * j_\varepsilon - f_i\|_{L^2(U)} &\le \|\lambda^d f_i^\lambda * j_\varepsilon - f_i^\lambda * j_\varepsilon\|_{L^2(U)} + \|f_i^\lambda * j_\varepsilon - f_i\|_{L^2(U)} \\ &\ll \delta + \|f_i^\lambda * j_\varepsilon - f_i * j_\varepsilon\|_{L^2(U)} + \|f_i * j_\varepsilon - f_i\|_{L^2(U)} \\ &= \|(f_i^\lambda - f_i) * j_\varepsilon\|_{L^2(U)} + \|f_i * j_\varepsilon - f_i\|_{L^2(U)} \\ &< \|f_i^\lambda - f_i\|_{L^2(U)} + \delta < 2\delta\end{aligned}$$

whenever λ is enough close to 1. This concludes the exercise.