D-MATH Functional analysis II Spring 2019
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Solutions for exercise sheet 2

1. We know by definition of the weak derivative of f that

$$\int_U (\boldsymbol{\partial}_\alpha f)\psi \, \mathrm{d}x = (-1)^{\|\alpha\|_1} \int_U f \boldsymbol{\partial}_\alpha \psi \, \mathrm{d}x$$

for all $\psi \in C_c^\infty(U)$. Now pick a sequence $(\psi_k)_k$ converging to g in $H_0^k(U)$. In particular, $\psi_k \to g$ and $\boldsymbol{\partial}_\alpha \psi \to \boldsymbol{\partial}_\alpha g$ in $L^2(U)$. Applying the above equality for $\psi = \psi_k$ and taking the limit for $k \to \infty$ we obtain the claim.

To answer the additional question, note that for example $g = 1 \in H^k(U)$ does not satisfy the equality for all $f \in H^k(U)$ when $U = (0, 1)$.

2. Let $U = (0, 1)$, $V = (0, \frac{1}{2})$ and $k = 1$. Consider the function

$$f = \mathbb{1}_V : x \mapsto \begin{cases} 1 & \text{if } x \in V \\ 0 & \text{else} \end{cases}$$

which is the extension of the function $1 \in H^1(V)$. We will show that $f \notin H^1(U)$. Suppose the contrary and let $f' \in L^2(U)$ be the weak derivative. If $\psi \in C_c^\infty(V)$ then

$$0 = \int_U \psi' \, \mathrm{d}x = \int_U \psi' f \, \mathrm{d}x = -\int_U \psi f' \, \mathrm{d}x$$

which implies that $f'|_V = 0$. Similarly, if $\psi \in C_c^\infty((\frac{1}{2}, 1))$ then

$$0 = \int_{(\frac{1}{2},1)} \psi' f \, \mathrm{d}x = -\int_U \psi f' \, \mathrm{d}x$$

shows that $f|_{(\frac{1}{2},1)} = 0$ and thus $f' = 0$. If we let $\psi \in C_c^\infty(U)$ be arbitrary, this shows that

$$\psi(\tfrac{1}{2}) = \int_V \psi' \, \mathrm{d}x = \int_U \psi' f \, \mathrm{d}x = -\int_U \psi f' \, \mathrm{d}x = 0$$

which is a contradiction.

3. **a)** We need to check that the denominator is non-zero first. If $t \in \mathbb{R}$ is such that $\psi_1(t) + \psi_1(-1-t) = 0$ then $\psi_1(t) = 0$ (i.e. $t \geq 0$) and $\psi_1(-1-t) = 0$ (i.e. $-1-t \geq 0$ which means $t \leq -1$). This is clearly impossible. Smoothness follows thus from smoothness of ψ_1 implies smoothness of ψ_2 by Leibniz' rule.

b) Again, smoothness follows from Leibniz' rule. Let us thus check the remaining properties. If $t \leq -2$ then $t+2 \leq 0$ or in other words $-(t+2) \geq 0$ which implies that $\psi_2(-2-t) = 0$ and thus $\psi_3(t) = 0$. For $t \geq 2$ one argues similarly. If $t \in [-1, 1]$, we have that $-2-t \leq -2+1 = -1$ and $t-2 \leq 1-2 = -1$ which implies that $\psi_2(-2-t) = \psi_2(t-2) = 1$ and thus $\psi_3(t) = 1$.

c) Let $x \in K$ and choose $r > 0$ so that $\overline{B_{2r}(0)} \subset U$. We then define

$$\psi_x : y \in U \mapsto \psi_3(\tfrac{1}{r}\|y - x\|).$$

If $y \in B_r(0)$ then $\|y-x\| < r$ so that $\frac{1}{r}\|y-x\| < 1$ which implies that $\psi_x(y) = 1$ by part b). Similarly, $y \notin \overline{B_{2r}(0)}$ implies $\frac{1}{r}\|y-x\| > 2$ and thus $\psi_x(y) = 0$.

d) For every $x \in K$ let $\psi_x \in C_c^\infty(U)$ be the function found in c) and let U_x be an open set containing x with $\psi_x|_{U_x} \equiv 1$. Then $\{U_x : x \in K\}$ forms an open cover of K so let $x_1, \ldots, x_m \in K$ be such that $U_{x_1} \cup \ldots U_{x_m} \supset K$ by compactness of the subset K. We define

$$\tilde{\phi} = \sum_{i=1}^{m} \psi_{x_i} \in C_c^\infty(U).$$

If $x \in K$ then $\tilde{\phi}(x) \geq 1$ as $x \in U_{x_i}$ for some i which implies $\psi_{x_i}(x) = 1$. We let

$$\phi : x \in U \mapsto \psi_2(-\tilde{\phi}(x)).$$

We have proven above that $\tilde{\phi}(x) \geq 1$ for $x \in K$ which yields $-\tilde{\phi}(x) \leq -1$ and thus $\phi(x) = 1$ in this case. Notice also that $\phi(x) = 0$ if $\tilde{\phi}(x) = 0$ and thus $\operatorname{supp}(\phi) \subset \operatorname{supp}(\tilde{\phi}) \subset U$. This concludes part d).

4. **a)** Let $\chi \in C_c^\infty(U)$ with $\chi|_K \equiv 1$ where $K = \operatorname{supp}(f)$. Choose also an open subset $V \subset U$ with $K \subset V \subset \overline{V} \subset U$. From Lemma 5.36 we know that $(\chi f)|_V \in H_0^k(V)$. But $f = \chi f$ and so $f \in H_0^k(U)$ (look at Exercise 1, Sheet 1).

b) Let us first prove that $f \in H^1(U)$. By the hint, it suffices to show that f and its weak derivatives are in $L^2(U)$. Note that in polar coordinates

$$\begin{aligned}\int_U |f(x)|^2 \,\mathrm{d}x &= \int_0^{2\pi} \int_0^1 (1-r)^{2\alpha} r \,\mathrm{d}r \,\mathrm{d}\varphi \\ &= 2\pi \int_0^1 (1-r)^{2\alpha} r \,\mathrm{d}r < \infty.\end{aligned}$$

We want to apply a similar argument for the weak derivatives of f. First this (compare Exercise 3, Sheet 1) note that we might as well only show that

$$\|\boldsymbol{\partial}_{e_1} f\|_{L^2}^2 + \|\boldsymbol{\partial}_{e_2} f\|_{L^2}^2 < \infty$$

We have

$$\partial_{e_i} f(x) = \alpha(1 - \|x\|)^{\alpha-1} \tfrac{1}{2\|x\|} 2x_i$$

and so

$$\begin{aligned}\|\boldsymbol{\partial}_{e_1} f\|_{L^2}^2 + \|\boldsymbol{\partial}_{e_2} f\|_{L^2}^2 &= \int_U \alpha^2 (1 - \|x\|)^{2(\alpha-1)} \,\mathrm{d}x = 2\pi \int_0^1 (1-r)^{2\alpha-2} r \,\mathrm{d}r \\ &= 2\pi \int_0^1 s^{2\alpha-2}(1-s)\,\mathrm{d}s.\end{aligned}$$

Splitting the last integral into two terms and recalling that $\alpha > \frac{1}{2}$, either of the two integrals is finite and so $f \in H^1(U)$.

Now let f_λ be defined as in the hint and note that by part a) and an analogous argument as in the first claim of this part of the exercise, $f_\lambda \in H_0^1(U)$. We claim that $f_\lambda \to f$ as $\lambda \to 1$ proving that $f \in H_0^1(U)$ as $H_0^1(U) \subset H^1(U)$ is closed.

To see this, we define the annulus $U_\lambda = \{x \in U : 1 - \|x\| < \lambda\}$ and compute

$$\begin{aligned}\int_U |f_\lambda(x) - f(x)|^2 \,\mathrm{d}x &= \int_{U_\lambda} |f(x)|^2 \,\mathrm{d}x + \int_{B_\lambda(0)} |f_\lambda(x) - f(x)|^2 \,\mathrm{d}x \\ &= \int_{U_\lambda} |f(x)|^2 \,\mathrm{d}x + \int_{B_\lambda(0)} |f(\lambda x) - f(x)|^2 \,\mathrm{d}x.\end{aligned}$$

The first integral goes to zero as $\lambda \to 1$ since $f \in L^2(U)$. The latter integral goes to zero for any continuous function of compact support in U_λ by uniform continuity. This can be extended to L^2-functions. Therefore, $\|f_\lambda - f\|_{L^2(U)} \to 0$.

We proceed similarly for the weak derivatives. Note that for almost every $x \in U$

$$\boldsymbol{\partial}_{e_i} f_\lambda(x) = \begin{cases} 0 & \text{if } x \in U_\lambda \\ \lambda \boldsymbol{\partial}_{e_i} f(\lambda x) & \text{if } x \notin U_\lambda \end{cases}.$$

Thus,

$$\begin{aligned}\int_U |\boldsymbol{\partial}_{e_i} f_\lambda(x) - \boldsymbol{\partial}_{e_i} f(x)|^2 \,\mathrm{d}x = \int_{U_\lambda} &|\boldsymbol{\partial}_{e_i} f(x)|^2 \,\mathrm{d}x \\ &+ \int_{B_\lambda(0)} |\lambda \boldsymbol{\partial}_{e_i} f(\lambda x) - \boldsymbol{\partial}_{e_i} f(x)|^2 \,\mathrm{d}x.\end{aligned}$$

Again, the first integral goes to zero as $\boldsymbol{\partial}_{e_i} f \in L^2(U)$. Taking care of the extra factor of λ (replace it first by one using the triangle inequality) one can use the same argument as above to conclude.

c) Note that $f|_{\partial B_{1-\delta}(0)} \equiv (1-(1-\delta))^\alpha = \delta^\alpha$ and therefore

$$\|f|_{\partial B_{1-\delta}(0)}\|_{L^2} = \delta^\alpha.$$

The analogous statement is true if one considers arcs A_δ (of angle independent of δ) instead of the whole circles. We consider such arcs A_δ where the angles are small enough for Proposition 5.33 to apply. In particular, there is a constant c independent of δ such that $\|f|_{A_\delta}\|_{L^2} = c\delta^\alpha$. Proposition 5.33 states that

$$c\delta^\alpha = \|f|_{A_\delta}\|_{L^2} \ll \sqrt{\delta}\|f\|_{H^1}.$$

Since this holds for all $\alpha > \frac{1}{2}$, the exponent $\frac{1}{2}$ on the right hand side cannot be improved.

5. a) Suppose that $H_0^k(Z) = H^k(U)$. Since $\mathbb{1}_U \in H^k(U)$, this implies that $\mathbb{1}_U \in H_0^k(U)$ (trivially). From Exercise 1 we conclude that for any $f \in H^k(U)$

$$\int_U \boldsymbol{\partial}_\alpha f \,\mathrm{d}x = \int_U \boldsymbol{\partial}_\alpha f \mathbb{1}_U \,\mathrm{d}x = (-1)^{\|\alpha\|_1} \int_U \psi \boldsymbol{\partial}_\alpha \mathbb{1}_U \,\mathrm{d}x = 0.$$

We claim that this is impossible. To this end, we pick a function $f \in H^k(U)$ with positive weak derivative in the α-direction. This contradicts the above equality. To simplify the exposition and since U is bounded, we may assume that that $U \subset \{x \in \mathbb{R}^d : x_i \leq -1 \text{ for all } i\}$. (One can reduce to this by shifting the function f defined below). A quick calculation shows that $x \mapsto \mathrm{e}^{-\frac{1}{2}\|x\|^2}$ is in $H^k(\mathbb{R}^d)$ and thus so is its restriction f to U. Note that

$$\partial_\alpha f(x) = (-x_1)^{\alpha_1} \cdots (-x_d)^{\alpha_d} \mathrm{e}^{-\|x\|^2} \geq \mathrm{e}^{\|x\|^2} > 0$$

by the way we placed U. This yields a contradiction to $\int_U \boldsymbol{\partial}_\alpha f \,\mathrm{d}x = 0$ and hence we are done.

b) Let $f \in C^\infty(\mathbb{R}^d) \cap H^k(\mathbb{R}^d)$ and let $\varepsilon > 0$. Choose $R \geq 1$ such that

$$\|\boldsymbol{\partial}_\alpha f\|_{L^2(\mathbb{R}^d \setminus B_R(0))} < \varepsilon$$

for all α with $\|\alpha\|_1 \leq k$.

We first construct a nice function $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\psi|_{B_R(0)} \equiv 1$. To do this, consider

$$\psi : x \in \mathbb{R}^d \mapsto \psi_3(\tfrac{1}{R}\|x\|)$$

where ψ_3 is as in Exercise 3. If $x \in B_R(0)$ we have $\frac{1}{R}\|x\| < 1$ and so $\psi(x) = 1$. The function ψ is nice in the following aspect: for any α we have that $\partial_\alpha \psi(x)$ is of the form $\frac{1}{R^{\|\alpha\|_1}}$ times a function bounded independently of R (as a calculation shows). Since $R \geq 1$, this shows that for any α

$$\|\partial_\alpha \psi\|_\infty \ll 1$$

where the implicit constant is absolute.

Now consider the function $\psi f \in C_c^\infty(\mathbb{R}^d)$. We then have for any α

$$\begin{aligned}\|\partial_\alpha f - \partial_\alpha(\psi f)\|_{L^2(\mathbb{R}^d)} &= \|\partial_\alpha f - \partial_\alpha(\psi f)\|_{L^2(U_R)} \\ &\leq \|\partial_\alpha f\|_{L^2(U_R)} + \|\partial_\alpha(\psi f)\|_{L^2(U_R)}\end{aligned}$$

where $U_R = \mathbb{R}^d \setminus B_R(0)$. By Leibniz' rule the weak derivative $\boldsymbol{\partial}_\alpha(\psi f)$ is a sum over terms of the form $c_{\beta,\gamma}\boldsymbol{\partial}_\beta f \boldsymbol{\partial}_\gamma \psi$ for $\|\gamma\|_1, \|\beta\|_1 \leq \|\alpha\|_1$. Therefore, by the triangle inequality

$$\|\boldsymbol{\partial}_\alpha(\psi f)\|_{L^2(U_R)} \ll \max_{\|\beta\|_1 \leq \|\alpha\|_1} \|\boldsymbol{\partial}_\beta f\|_{L^2(U_R)} \leq \varepsilon.$$

Combining this with the above

$$\|\partial_\alpha f - \partial_\alpha(\psi f)\|_{L^2(\mathbb{R}^d)} \leq 2\varepsilon$$

for any α which is what we desired.

6. a) We only need to show that any $f \in C^\infty(\mathbb{R}_{>0}) \cap H^1(\mathbb{R}_{>0})$ can be approximated by functions as in the claim. Let $\varepsilon > 0$ and fix $x_0 \in (0,1)$ such that

$$\int_0^{x_0} |f(x)|^2 \,\mathrm{d}x < \varepsilon^2 \quad \text{and} \quad \int_0^{x_0} |f'(x)|^2 \,\mathrm{d}x < \varepsilon^2.$$

In particular, if $a > 0$ is fixed (to be chosen later) and if $A_f \subset [0, x_0]$ denotes the (measurable) set of points x with $|f(x)|^2 > a\varepsilon^2$, then the measure of A_f is bounded by

$$\frac{1}{a\varepsilon^2}\int_0^{x_0} |f(x)|^2 < \frac{1}{a}$$

A set $A_{f'}$ is defined analogously and one also obtains that $A_{f'}$ has measure $< \frac{1}{a}$. We choose $a = 4$ and obtain that $A_f \cup A_{f'}$ has measure $< \frac{1}{2}$. We may therefore choose a point $x \notin A_f \cup A_{f'}$ with $x \geq \frac{1}{2}x_0$ (which is irrelevant for us) and

$$|f(x)| < 2\varepsilon \quad \text{and} \quad |f'(x)| < 2\varepsilon.$$

We replace x_0 by x. Define $\psi \in C^1(\mathbb{R}_{\geq 0})$ by setting for any $t \in \mathbb{R}_{>0}$

$$\psi(t) = \begin{cases} f(t) & \text{if } t \geq x \\ (\frac{t^2}{2} - \frac{x^2}{2})f'(x) + f(x) & \text{else} \end{cases}$$

where the latter is a polynomial interpolation with value $f(x)$ at x, derivative $f'(x)$ at x and derivative 0 at 0. By the very definition of ψ

$$\|f-\psi\|^2_{L^2(\mathbb{R}_{>0})} = \|f-\psi\|^2_{L^2([0,x])} \leq \|f\|^2_{L^2([0,x])} + \|\psi\|^2_{L^2([0,x])} \\ < \varepsilon^2 + \|\psi\|^2_{L^2([0,x])}$$

and similarly $\|f-\psi\|^2_{L^2(\mathbb{R}_{>0})} < \varepsilon + \|\psi'\|^2_{L^2([0,x])}$ so it remains to bound the L^2-norm of ψ and ψ' on $[0,x]$. This is done rather directly via

$$\|\psi\|^2_{L^2([0,x])} \ll \int_0^x (\tfrac{t^2}{2} - \tfrac{x^2}{2})^2 |f'(x)|^2 + |f(x)|^2 \,\mathrm{d}t \\ \ll \varepsilon^2 \int_0^x (\tfrac{t^2}{2} - \tfrac{x^2}{2})^2 + 1 \,\mathrm{d}t \ll \varepsilon^2$$

and similarly for the derivative. Note that $\psi, \psi' \in L^2(\mathbb{R}_{>0})$ implies that $\psi \in H^1(\mathbb{R}_{>0})$ by Exercise 6, Sheet 1.

b) By part a) it suffices to define ext on $f \in C^1(\mathbb{R}_{\geq 0}) \cap H^1(\mathbb{R}_{>0})$ with $f'(0) = 0$ which we do by

$$\mathrm{ext}(f)(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$$

for every $x \in \mathbb{R}$. Since $f'(0) = 0$, $\mathrm{ext}(f)$ is continuously differentiable with derivative $f'(x)$ for $x \in \mathbb{R}_{\geq 0}$ and $-f'(-x)$ for $x < 0$. Also,

$$\|\mathrm{ext}(f)\|^2_{L^2(\mathbb{R})} = 2\|f\|^2_{L^2(\mathbb{R}_{>0})} \text{ and } \|\mathrm{ext}(f)'\|^2_{L^2(\mathbb{R})} = 2\|f'\|^2_{L^2(\mathbb{R}_{>0})}.$$

This shows that

$$\mathrm{ext} : \{f \in C^1(\mathbb{R}_{\geq 0}) \cap H^1(\mathbb{R}_{>0}) : f'(0) = 0\} \to H^1(\mathbb{R})$$

is bounded which yields by extension a bounded operator

$$\mathrm{ext} : H^1(\mathbb{R}_{>0}) \to H^1(\mathbb{R}).$$

The composition ext with the restriction operator $H^1(\mathbb{R}) \to H^1(\mathbb{R}_{>0})$ is by construction the identity on the subspace defined in a) and does everywhere by density. This completes the exercise.