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Solutions for exercise sheet 3

1. Let $f \in H^1(U)$ and let $\phi \in C_c^\infty(U)$. We compute by partial integration

$$\begin{aligned}\langle f, \phi \rangle_1 &= \sum_{j=1}^d \int_U \boldsymbol{\partial}_{e_j} f(x) \partial_{e_j} \phi(x)\,\mathrm{d}x = -\sum_{j=1}^d \int_U f(x) \partial_{e_j}^2 \phi(x)\,\mathrm{d}x \\ &= -\int_U f(x) \sum_{j=1}^d \partial_{e_j}^2 \phi(x)\,\mathrm{d}x = -\int_U f(x) \triangle \phi(x)\,\mathrm{d}x.\end{aligned}$$

2. **a)** Let $f \in H_0^1(U)$ be weakly harmonic. Then by Exercise 1 we have that $\langle f, \phi \rangle_1 = 0$ for all $\phi \in C_c^\infty(U)$. Since $C_c^\infty(U)$ is dense in $H_0^1(U)$ we deduce that $\langle f, g \rangle_1 = 0$ for all $g \in H_0^1(U)$. We proved however in Lemma 5.41 that $\langle \cdot, \cdot \rangle_1$ defines an inner product, which thus implies that $f = 0$.

b) We may proceed exactly as in the proof of Proposition 5.42 to obtain such a decomposition. Notice first though that the decomposition is unique: if

$$f = g_1 + v_1 = g_2 + v_2$$

for $g_1, g_2 \in H^1(U)$ weakly harmonic and $v_1, v_2 \in H_0^1(U)$. Then $g = g_1 - g_2 = v_2 - v_1 \in H_0^1(U)$ is weakly harmonic and therefore $g = 0$ by part a) which means $g_1 = g_2$ and $v_1 = v_2$.

Let us now prove the existence. The linear functional $\ell : g \in H_0^1(U) \mapsto \langle f, g \rangle_1$ is bounded and hence there exists $v \in H_0^1(U)$ which $\langle f, g \rangle_1 = \langle v, g \rangle_1$ for all $g \in H_0^1(U)$. In particular,

$$0 = \langle f - v, \phi \rangle_1$$

for all $\phi \in C_c^\infty(U)$ which is to say that $f - v$ is weakly harmonic as desired.

3. **a)** This is immediate as an open cover of a compact metric space has a Lebesgue number. We choose r to be $\frac{1}{4}$ of this number.

b) See Exercise 3c), Sheet 2 and its solution.

c) Let $x_1, \dots, x_k \in K$ with $X \subset \bigcup_{i=1}^k B_r(x_i)$ and let $\tilde{\chi}_1, \dots, \tilde{\chi}_k$ be the functions defined found in b) for the points $x_1, \dots, x_k \in K$. Let

$$U = \bigcup_{i=1}^k \operatorname{supp}(\tilde{\chi}_i)^\circ$$

which contains K by construction. By Exercise 3, Sheet 2 there exists a function in $C_c^\infty(U)$ which is equal to 1 on K. We let $\psi \in C^\infty(\mathbb{R}^d)$ be 1 minus that function so that $\psi \equiv 0$ on K and 1 outside of U. Now define for $i = 1, \dots, k$

$$\chi_i : x \in \mathbb{R}^d \mapsto \frac{\tilde{\chi}_i(x)}{\psi(x) + \sum_{j=1}^k \tilde{\chi}_j(x)}.$$

Note that by construction the denominator is nowhere vanishing. Also the functions χ_i sum to 1 on K since ψ is zero on K. This concludes the claim of the exercise.

4. **a)** We will use induction over $\ell \in \{0, \dots, k\}$ showing that the claim holds for all α with $\|\alpha\|_1 = \ell$. For $\ell = 0$ there is nothing to show. So suppose that the claim holds for all $\ell' < \ell$ and let α satisfy $\|\alpha\|_1 = \ell$. Assume without loss of generality that $\alpha_1 \neq 0$ and set $\alpha' = \alpha - e_1$. Then

$$\partial_{\alpha'}(f \circ \Phi) = \sum_{\|\beta\|_1 \leq \ell - 1} g_{\alpha',\beta}(\partial_\beta f) \circ \Phi$$

by the induction hypothesis. Therefore, by the Leibniz rule

$$\begin{aligned}\partial_\alpha(f \circ \Phi) &= \partial_{e_1}(\partial_{\alpha'}(f \circ \Phi)) \\ &= \sum_{\|\beta\|_1 \leq \ell - 1} \partial_{e_1}\Big(g_{\alpha',\beta}(\partial_\beta f) \circ \Phi\Big) \\ &= \sum_{\|\beta\|_1 \leq \ell - 1} \partial_{e_1} g_{\alpha',\beta}(\partial_\beta f) \circ \Phi + g_{\alpha',\beta}\partial_{e_1}((\partial_\beta f) \circ \Phi).\end{aligned}$$

Notice that the sum over the first terms already has the desired form. We have by the chain rule

$$\partial_{e_1}((\partial_\beta f) \circ \Phi) = \sum_{k=1}^d \partial_{e_k}\partial_\beta f \circ \Phi \cdot \partial_{e_1}\Phi_k.$$

Plugging this into the above expression we deduce the claim.

b) Since $\tilde{V}$ contains the closure of V and the latter is compact, we have that

$$\sum_{\|\beta\|_1 \leq \|\alpha\|_1 \leq k} \|g_{\alpha,\beta}\|_\infty \leq M$$

for some $M > 0$. Therefore, we have for any $f \in C^\infty(V) \cap H^k(V)$ and any α

$$\begin{aligned}\|\partial_\alpha(f \circ \Phi)\|_{L^2(U)} &\leq \sum_{\|\beta\|_1 \leq \|\alpha\|_1} \|g_{\alpha,\beta}(\partial_\beta f) \circ \Phi\|_{L^2(U)} \\ &\leq M \sum_{\|\beta\|_1 \leq \|\alpha\|_1} \|(\partial_\beta f) \circ \Phi\|_{L^2(U)}\end{aligned}$$

We can increase M if necessary so that $\|\det(\mathrm{D}\Phi^{-1})\|_\infty \leq M$. Thus, by substitution

$$\|\partial_\alpha(f \circ \Phi)\|_{L^2(U)} \leq M^2 \sum_{\|\beta\|_1 \leq \|\alpha\|_1} \|\partial_\beta f\|_{L^2(V)} \ll \|f\|_{H^k(V)}$$

This shows that the operator

$$H^k(V) \ni f \mapsto f \circ \Phi \in H^k(U)$$

is bounded. By the same argument, its inverse

$$H^k(U) \ni f \mapsto f \circ \Phi^{-1} \in H^k(V)$$

is bounded and hence we conclude.

Assume now that Φ is an isometry and write $\Phi(x) = Rx + a$ as in the statement of the exercise. In particular, $\mathrm{D}\Phi = R$ (that is, $\partial_{e_j}\Phi_\ell = R_{\ell j}$) and all higher derivatives vanish. For any $f \in C^\infty(V)$ we thus have

$$\begin{aligned}\partial_{e_j}(f \circ \Phi) &= \sum_\ell (\partial_{e_\ell} f) \circ \Phi \cdot \partial_{e_j}\Phi_\ell \\ &= \sum_\ell R_{\ell j}(\partial_{e_\ell} f) \circ \Phi.\end{aligned}$$

Let us first prove the statement for $k = 1$. So let $f \in C^\infty(V) \cap H^1(V)$ and first note that $\|f \circ \Phi\|_{L^2(U)} = \|f\|_{L^2(V)}$ by substitution. We compute for any j

$$\begin{aligned}\|\partial_{e_j}(f \circ \Phi)\|^2_{L^2(U)} &= \int_U |\partial_{e_j}(f \circ \Phi)|^2 \,\mathrm{d}x \\ &= \sum_{\ell_1,\ell_2} \int_U R_{\ell_1 j} R_{\ell_2 j} (\partial_{e_{\ell_1}} f) \circ \Phi \cdot (\partial_{e_{\ell_2}} f) \circ \Phi\end{aligned}$$

and summing over j we get

$$\begin{aligned}\sum_j \|\partial_{e_j}(f\circ\Phi)\|_{L^2(U)}^2 &= \sum_{\ell_1,\ell_2}\int_U (\partial_{e_{\ell_1}} f)\circ\Phi\cdot(\partial_{e_{\ell_2}} f)\circ\Phi \sum_j R_{\ell_1 j}R_{\ell_2 j} \\ &= \sum_{\ell_1,\ell_2}\int_U (\partial_{e_{\ell_1}} f)\circ\Phi\cdot(\partial_{e_{\ell_2}} f)\circ\Phi\delta_{\ell_1,\ell_2} \\ &= \sum_{\ell_1}\|(\partial_{e_{\ell_1}} f)\circ\Phi\|_{L^2(U)}^2\end{aligned}$$

where we crucially used orthogonality of R. By substitution we obtain

$$\sum_j \|\partial_{e_j}(f\circ\Phi)\|_{L^2(U)}^2 = \sum_{\ell_1}\|\partial_{e_{\ell_1}} f\|_{L^2(V)}^2$$

which proves that $H^1(V)\ni f\mapsto f\circ\Phi\in H^1(U)$ is an isometry. To get the case $k>1$ one can apply the equality

$$\sum_j \|\partial_{e_j}(f\circ\Phi)\|_{L^2(U)}^2 = \sum_{\ell_1}\|(\partial_{e_{\ell_1}} f)\circ\Phi\|_{L^2(U)}^2$$

inductively to the derivatives of f instead of f to obtain the claim.

5. We review the proof of Theorem 5.34. Let $K\subset U$ be compact and let $\chi\in C_c^\infty(U)$ with $\chi|_K\equiv 1$. As in the proof of Theorem 5.34 each of the operators in the chain

$$H^k(U)\overset{P\circ M_\chi}{\to} H^k(\mathbb{T}_R^d)\overset{\iota}{\to} C(\mathbb{T}_R^d)\overset{|_K}{\to} C(K)$$

is bounded and their composition is exactly the restriction to K. This proves the claim.

6. Let $M=\|f\|_\infty$.

a) Since f is bounded, we have for $x+iy\in\mathbb{H}$

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{y}{(x-t)^2+y^2}|f(t)|\,\mathrm{d}t &\le M\int_{-\infty}^{\infty}\frac{y}{(x-t)^2+y^2}\,\mathrm{d}t \\ &= M\int_{-\infty}^{\infty}\frac{y}{t^2+y^2}\,\mathrm{d}t \\ &= M\int_{-\infty}^{\infty}\frac{1}{1+s^2}\,\mathrm{d}s<\infty\end{aligned}$$

where we set $s=\frac{t}{y}$. This shows that the integral defining $\tilde{f}$ is absolutely convergent and hence convergent. Note that this proof gives a uniform upper bound and hence also gives the second statement in 6c).

For $N \in \mathbb{N}$ define

$$\tilde{f}_N(x+\mathrm{i}y) = \int_{-N}^{N} \frac{y}{(x-t)^2+y^2} f(t)\,\mathrm{d}t.$$

By the above estimate

$$|\tilde{f}(x+\mathrm{i}y) - \tilde{f}_N(x+\mathrm{i}y)| \le 2M \int_N^\infty \frac{1}{s^2}\,\mathrm{d}s = 2M\frac{1}{N}$$

which shows that $\tilde{f}_N \to \tilde{f}$ uniformly. The continuity of $\tilde{f}$ thus follows from the continuity of $\tilde{f}_N$, the latter being a standard statement from real analysis.

b) The first claim follows directly from the computation

$$\begin{aligned}
\tilde{f}(x+\mathrm{i}y) &= \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} f(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \frac{1}{y}\frac{1}{(\frac{x-t}{y})^2+1} f(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \tfrac{1}{y}\chi\big(\tfrac{x-t}{y}\big) f(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \chi_y(x-t) f(t)\,\mathrm{d}t.
\end{aligned}$$

Note that for any $a > 0$

$$\begin{aligned}
\int_a^\infty \chi_y(s)\,\mathrm{d}s = \int_a^\infty \tfrac{1}{y}\chi\big(\tfrac{s}{y}\big)\,\mathrm{d}s &= \int_{\frac{a}{y}}^\infty \chi(u)\,\mathrm{d}u \\
&\le \int_{\frac{a}{y}}^\infty \frac{1}{u}\,\mathrm{d}u = \frac{y}{a}.
\end{aligned}$$

by substituting $\frac{s}{y} = u$. Note also that $\int_{-\infty}^{-a} \chi_y(s)\,\mathrm{d}s = \int_a^\infty \chi_y(s)\,\mathrm{d}s$.

Let $x_0 \in \mathbb{R}$ and let $\varepsilon > 0$. Let $\delta > 0$ be as in the definition of continuity of f at x_0. Suppose that $x + \mathrm{i}y \in \mathbb{H}^2$ is $\min(\frac{\delta}{2}, \frac{\delta\cdot\varepsilon}{2M})$-close to x_0. Note that for any $t \in (-\frac{\delta}{2}, \frac{\delta}{2})$ we have $|(x-t)-x_0| < \delta$. We then estimate

$$\begin{aligned}
|\tilde{f}(x+\mathrm{i}y) - f(x_0)| &\le \int_{-\infty}^{\infty} \chi_y(t)|f(x-t)-f(x_0)|\,\mathrm{d}t \\
&\le 2M\int_\delta^\infty \chi_y(t)\,\mathrm{d}t + \int_{-\delta}^{\delta} \chi_y(t)|f(x-t)-f(x_0)|\,\mathrm{d}t \\
&\le 2M\frac{y}{\delta} + \varepsilon\int_{-\delta}^{\delta}\chi_y(t)\,\mathrm{d}t \le 2\varepsilon
\end{aligned}$$

which concludes the claim.

c) As mentioned, boundedness was already proven. To show that $\tilde{f}$ is smooth, let us show that

$$g : (x, y) \mapsto \chi_y(x) = \tfrac{1}{y}\chi(\tfrac{x}{y}) = \frac{y}{x^2 + y^2}$$

is smooth. For this, observe first that $\chi : t \mapsto \frac{1}{1+t^2}$ is smooth and the n-th derivative $\chi^{(n)}$ of χ is of the form $\frac{p_n(t)}{(1+t^2)^{n+1}}$ where p_n is a polynomial of degree n (explicitly computable, but we don't need such an expression here in general). In particular, $\chi^{(n)}$ is in $L^1(\mathbb{R})$ for every n. The smoothness claim for the map g is now immediate as an expression in smooth functions.

Let us also note that

$$\begin{aligned}
\partial_x g(x, y) &= \frac{-2xy}{(x^2 + y^2)^2}, \\
\partial_y g(x, y) &= \frac{1}{x^2 + y^2} - \frac{2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \\
\partial_x^2 g(x, y) &= \frac{-2y}{(x^2 + y^2)^2} - 2\frac{-2xy \cdot 2x}{(x^2 + y^2)^3} = -2y\frac{-3x^2 + y^2}{(x^2 + y^2)^3} \\
\partial_y^2 g(x, y) &= \frac{-2y}{(x^2 + y^2)^2} - 2\frac{(x^2 - y^2)2y}{(x^2 + y^2)^3} = -2y\frac{3x^2 - y^2}{(x^2 + y^2)^3}
\end{aligned}$$

which gives in particular that $\triangle g = 0$. This (at least intuitively) explains why we can expect $\tilde{f}$ to be harmonic. Recall that

$$\tilde{f}(x + \mathrm{i}y) = \int_{-\infty}^{\infty} g(x - t, y) f(t)\,\mathrm{d}t.$$

CLAIM: $\tilde{f}$ is smooth and

$$\partial_\alpha \tilde{f}(x + \mathrm{i}y) = \int_{-\infty}^{\infty} \partial_\alpha g(x - t, y) f(t)\,\mathrm{d}t$$

Notice that by the above calculations the claim is everything we need to prove. To prove the claim (since smoothness is a local property) we may assume that $y > y_0$ for some fixed $y_0 > 0$. We claim that this implies that all partial derivatives of g are uniformly continuous. Recall that the n-th derivative $\chi^{(n)}$ of χ is of the form $\frac{p_n(t)}{(1+t^2)^{n+1}}$ where p_n is a polynomial of degree n and is in particular bounded. By induction, any partial derivative of g is a finite sum of terms of the form $y^{-m_1}\chi^{(m_2)}(\frac{x}{y})$. In particular, we have that

$$\|\partial_\alpha g|_{\{x+\mathrm{i}y : y > y_0\}}\|_\infty < \infty$$

for any α. This implies (by considering the total derivative) that $\partial_\alpha g|_{\{x+\mathrm{i}y:y>y_0\}}$ is uniformly continuous for any α.

Using this uniform continuity one can show the claim by a simple argument from real analysis; we only sketch the argument here. For simplicity of notation (and only that) we show the claim for $\alpha = e_1$. Writing down the difference quotient for this partial derivative and using linearity we obtain an expression of the form

$$\int_{-\infty}^{\infty} \Big(\frac{g(x+h-t,y) - g(x-t,y)}{h} - \partial_{e_1} g(x-t,y) \Big) f(t) \, \mathrm{d}t. \qquad (1)$$

By the mean value theorem the difference quotient $\frac{g(x+h-t,y)-g(x-t,y)}{h}$ is equal to $\partial_{e_1} g(x-t+h\xi, y)$ for some $\xi \in [0,1]$. Since the point $(x-t+h\xi, y)$ is within distance h of $(x-t,y)$, we may choose h small enough so that

$$\frac{g(x+h-t,y) - g(x-t,y)}{h} \asymp_\varepsilon \partial_{e_1} g(x-t,y)$$

where ε is given at the beginning and the error is independent of x, t, y. We cannot use this argument immediately as we are integrating over a region of infinite measure. Instead, we first a compact interval I around x so that

$$\int_{\mathbb{R}\setminus I} |\partial_{e_1} g(x-t,y)| \, \mathrm{d}t \le \varepsilon$$

(by a similar argument as used above). Analyzing the step which used the intermediate value theorem more carefully, one sees that one can apply Taylor's theorem in the form

$$\frac{g(x+h-t,y) - g(x-t,y)}{h} = \partial_{e_1} g(x-t,y) + h R_h(x-t,y)$$

where

$$R_h(x-t,y) = \int_0^1 (1-s) \partial_{2e_1} g(x-t+sh, y) \, \mathrm{d}s.$$

One checks that $t \mapsto R_h(x-t,y)$ is L^1 with an upper bound to the L^1-norm independent of x and y. This shows that the integral over the difference quotient outside of I may be estimated in the same way as the integral over $\partial_{e_1} g(x-t,y)$. Therefore, the integral in (1) can be replaced by the integral over I which gives the claim by the argument sketched earlier.