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Solutions for exercise sheet 4

1. Assume first that $g \in H_0^1(U)$ and recall from Lemma 6.67 that the functions $|\lambda_n|^{-\frac{1}{2}} f_n$ form an orthonormal basis. We may thus write

$$g = \sum_{n=1}^{\infty} b_n |\lambda_n|^{-\frac{1}{2}} f_n$$

where $\|g\|_1^2 = \sum_{n=1}^{\infty} |b_n|^2$. Since $\iota : H_0^1(U) \to L^2(U)$ is bounded, we also have $g = \sum_{n=1}^{\infty} b_n |\lambda_n|^{-\frac{1}{2}} f_n$ in $L^2(U)$ and therefore $a_n = b_n |\lambda_n|^{-\frac{1}{2}}$. In particular,

$$\infty > \sum_{n=1}^{\infty} |b_n|^2 = \sum_{n=1}^{\infty} |a_n|^2 |\lambda_n|.$$

For the converse note that $\sum_{n=1}^{\infty} |a_n|^2 |\lambda_n| < \infty$ implies that the series

$$\sum_{n=1}^{\infty} \|a_n f_n\|_1^2 < \infty$$

and therefore, $\sum_{n=1}^{\infty} a_n f_n$ converges in $H_0^1(U)$. In particular, this converges holds when applying ι and thus the series must be equal to g. This shows that $g \in H_0^1(U)$.

2. By Weyl's law we have that the eigenvalue counting function $N_U(T)$ satisfies

$$\lim_{T \to \infty} \frac{N_U(T)}{T^{\frac{d}{2}}} = C_U$$

where $C_U = (2\pi)^{-d} \omega_d m(U)$. We now set $T = |\lambda_n|$ and obtain that

$$N_U(|\lambda_n|) = |\{m : |\lambda_m| \leq |\lambda_n|\}| \geq n$$

where equality might not occur due to multiplicities in the eigenvalues. Therefore, we obtain by taking a power in Weyl's law

$$\limsup_{n \to \infty} \frac{n^{\frac{2}{d}}}{|\lambda_n|} \leq \lim_{n \to \infty} \frac{N_U(|\lambda_n|)^{\frac{2}{d}}}{|\lambda_n|} = C_U^{\frac{2}{d}}$$

which proves one half of the desired statement.

For the other half, consider the adapted counting function

$$N'_U(T) = |\{m : |\lambda_m| < T\}|.$$

Then $N_U(T-1) \leq N'_U(T) \leq N_U(T)$ for any T. This implies that

$$\begin{aligned} C_U = \lim_{T\to\infty} \frac{N_U(T-1)}{(T-1)^{\frac{d}{2}}} &= \lim_{T\to\infty} \frac{N_U(T-1)}{T^{\frac{d}{2}}} \frac{T^{\frac{d}{2}}}{(T-1)^{\frac{d}{2}}} = \lim_{T\to\infty} \frac{N_U(T-1)}{T^{\frac{d}{2}}} \\ &\leq \liminf_{T\to\infty} \frac{N'_U(T)}{T^{\frac{d}{2}}} \leq \limsup \frac{N'_U(T)}{T^{\frac{d}{2}}} \leq \lim_{T\to\infty} \frac{N_U(T)}{T^{\frac{d}{2}}} = C_U \end{aligned}$$

and so Weyl's law holds also for $N'_U(T)$. Notice that

$$N'_U(|\lambda_n|) = |\{m : |\lambda_m| < |\lambda_n|\}| < n$$

and thus

$$\liminf_{n\to\infty} \frac{n^{\frac{2}{d}}}{|\lambda_n|} \geq \lim_{n\to\infty} \frac{N'_U(|\lambda_n|)^{\frac{2}{d}}}{|\lambda_n|} = C_U^{\frac{2}{d}}.$$

This finishes the proof.

3. **a)** One shows by direct computation that

$$\triangle f(x) = -(\lambda_1^2 + \ldots + \lambda_d^2) f(x).$$

Note that on boundary pieces of the form $V_{i,1} = \{x \in \partial U : x_i = 0\}$ for $i \in \{1, \ldots, d\}$ the function f vanishes independently of the values $\lambda_1, \ldots, \lambda_d$. It is therefore sufficient to characterize the set of tuples for which f vanishes identically on each set of the form $V_{i,2} = \{x \in \partial U : x_i = a_i\}$. Note that for any $x \in V_{i,2}$

$$f(x) = \sin(\lambda_i a_i) \prod_{j\neq i} \sin(\lambda_j x_j)$$

and thus $f|_{V_{i,2}} \equiv 0$ if and only if $\sin(\lambda_i a_i) = 0$. The latter is equivalent to $\lambda_i a_i$ being a multiple of π, that is,

$$\lambda_i = \frac{n_i}{\pi a_i}$$

for some $n_i \in \mathbb{N}$. In summary, requiring the above equality for every i is equivalent to f vanishing on ∂U.

b) This is contained in the proof of Proposition 6.65.

4. **a)** The proof of Proposition 6.65 for $R = 1$ shows that for any $T > 0$ we have

$$N(T) := N_{\mathbb{T}^d}(T) = |\mathbb{Z}^d \cap \overline{B_S^{\mathbb{R}^d}(0)}|$$

where $S = T^{\frac{1}{2}}(2\pi)^{-1}$. Therefore,

$$\left| N(T) - \mathrm{vol}(B_S^{\mathbb{R}^d}(0)) \right| \ll S^a \ll T^{\frac{a}{2}}.$$

b) As proven in Proposition 6.65, we have for any S

$$B_{S-\sqrt{d}}^{\mathbb{R}^d}(0) \subset (\mathbb{Z}^d \cap \overline{B_S^{\mathbb{R}^d}(0)}) + [-\tfrac{1}{2}, \tfrac{1}{2})^d \subset B_{S+\sqrt{d}}^{\mathbb{R}^d}(0)$$

where the volume of the expression in the middle is exactly

$$N'(S) = |\mathbb{Z}^d \cap \overline{B_S^{\mathbb{R}^d}(0)}|.$$

Therefore,

$$\begin{aligned} |N'(S) - \mathrm{vol}(B_S^{\mathbb{R}^d}(0))| &\le \mathrm{vol}(B_{S+\sqrt{d}}^{\mathbb{R}^d}(0)) - \mathrm{vol}(B_{S-\sqrt{d}}^{\mathbb{R}^d}(0)) \\ &= \omega_d((S+\sqrt{d})^d - (S-\sqrt{d})^d) \\ &= 2\omega_d\sqrt{d}S^{d-1} + O(S^{d-2}) \ll S^{d-1} \end{aligned}$$

as desired.

5. **a)** The statement that $g \in H^{k+2}(\mathbb{T}^d)$ is exactly the statement of Lemma 5.48. It remains to prove the claimed estimate. Note that for any $\alpha \neq 0$ with $\|\alpha\|_1 \le k+2$ we have

$$\|\boldsymbol{\partial}_\alpha g\|_{L^2}^2 = \sum_{n\in\mathbb{Z}} |c_n|^2 |(2\pi n)^\alpha|^2 \ll \sum_{n\in\mathbb{Z}} |c_n|^2 |n^\alpha|^2 \le \sum_{n\in\mathbb{Z}} |c_n|^2 \|n\|_2^{k+2} \ll \|u\|_{H^k(\mathbb{T}^d)}$$

where c_n are the Fourier coefficients of u. This proves the claim.

b) We will only consider the case $|\lambda| > 1$. Otherwise, one can apply an argument as in c) below. We know from Theorem 5.45 that $f \in H_{\mathrm{loc}}^{k+2}(U)$. Let $\chi \in C_c^\infty(U)$ and set $\Delta(\chi f) = u_1$. Let $\psi \in C_c^\infty(U)$ be such that $\psi \equiv 1$ on $\mathrm{supp}(\chi)$. The proof of Theorem 5.45 then shows that

$$\Delta P(\psi\chi f) = P(\psi u_1)$$

where we used notation from Lemma 5.36. Using a) for $\mathbb{T}_R$ instead of $\mathbb{T}$ (as luckily the proof is analogous) where R is chosen as in the proof of Theorem 5.45 we get

$$\begin{aligned}\|\chi f\|^2_{H^{k+2}(U)} = \|\psi\chi f\|^2_{H^{k+2}(U)} &= \|P(\psi\chi f)\|^2_{H^{k+2}(\mathbb{T}^d_R)} \\ &\ll \|P(\psi\chi f)\|^2_{L^2(\mathbb{T}^d_R)} + \|P(\psi u_1)\|^2_{H^k(\mathbb{T}^d_R)} \\ &= \|\chi f\|^2_{L^2(U)} + \|\psi u_1\|^2_{H^k(U)} \\ &\ll_\chi \|f\|^2_{L^2(U)} + \|\psi u_1\|^2_{H^k(U)}\end{aligned}$$

Now recall from Lemma 5.50 that

$$\begin{aligned}\psi u_1 = u_1 &= \chi(\triangle f) + (\triangle\chi) f + 2\sum_{j=1}^d (\partial_j\chi)(\partial_j f) \\ &= \lambda\chi f + (\triangle\chi) f + 2\sum_{j=1}^d (\partial_j\chi)(\partial_j f).\end{aligned}$$

This proves that

$$\begin{aligned}\|\psi u_1\|^2_{H^k(U)} &\ll \|\lambda\chi f\|^2_{H^k(U)} + \|(\triangle\chi) f\|^2_{H^k(U)} + \sum_{j=1}^d \|(\partial_j\chi)(\partial_j f)\|^2_{H^k(U)} \\ &\ll |\lambda|\,\|\chi f\|^2_{H^k(U)} + \|(\triangle\chi) f\|^2_{H^k(U)} + \sum_{j=1}^d \|(\partial_j\chi)(\partial_j f)\|^2_{H^k(U)}\end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}\|\chi f\|^2_{H^{k+2}(U)} \ll_\chi \|f\|^2_{L^2(U)} &+ |\lambda|\,\|\chi f\|^2_{H^k(U)} + \|(\triangle\chi) f\|^2_{H^k(U)} \\ &+ \sum_{j=1}^d \|(\partial_j\chi)(\partial_j f)\|^2_{H^k(U)}.\end{aligned}$$

We now claim that we can now proceed by induction for which we analyze the right hand side in this inequality. The term $\|f\|^2_{L^2(U)}$ is as we want it in the claim. For $\|\chi f\|^2_{H^k(U)}$ one can apply the same argument of course and similarly for $\|(\triangle\chi) f\|^2_{H^k(U)}$ as we still have $\triangle\chi \in C_c^\infty(U)$. For the remaining terms of the form $\|(\partial_j\chi)(\partial_j f)\|^2_{H^k(U)}$ note first that $\partial_j\chi \in C_c^\infty(U)$. Also, note that $f \in C^\infty(U)$ by elliptic regularity which implies $\partial_j f \in C^\infty(U)$. The latter is also an eigenfunction of $\triangle$ and hence we may also apply induction to this term.

It remains to explain why the claim holds for $k = 1$. But this is contained in Lemma 6.67.

c) By Exercise 5 on Sheet 3 we should choose $k \in \mathbb{N}$ with $\frac{d}{2} < k \leq \frac{d}{2} + 1$ which yields for any $\chi \in C_c^\infty(U)$

$$\|\chi f\|_{K,\infty} \ll_K \|\chi f\|_{H^k(U)}.$$

Choosing $\chi \in C_c^\infty(U)$ with $\chi|_K \equiv 1$ we deduce

$$\|f\|_{K,\infty} \ll_K \|\chi f\|_{H^k(U)}.$$

By part b) we have for $|\lambda| \geq 1$

$$\|\chi f\|_{H^k(U)} \ll |\lambda|^{\frac{k}{2}} \|f\|_{L^2(U)} \leq |\lambda|^{\frac{1}{2}(\frac{d}{2}+1)} \|f\|_{L^2(U)}$$

and thus the lemma is proven in this case. Note that the subspace V of $L^2(U)$ spanned by eigenfunctions of eigenvalue $|\lambda| < 1$ is finite-dimensional. Thus, $\|\cdot\|_{K,\infty}$ (which is bounded by a seminorm on V) can be bounded by a constant times $\|\cdot\|_{L^2(U)}$ on V (which is a norm). This proves the claim.

6. **a)** See Section 1.2.1.

b) Note first that indeed $u(\cdot, t) \in L^2(U)$ as

$$\|u(\cdot,t)\|_{L^2(U)}^2 = \sum_{n=1}^{\infty} |a_n^2| e^{2\lambda_n t} = \sum_{n=1}^{\infty} |a_n^2| e^{-2|\lambda_n| t} \leq e^{-|\lambda_1| t} \sum_{n=1}^{\infty} |a_n^2|.$$

It is worth it to remark here that this proves $u(\cdot, t) \to 0$ in L^2 as $t \to \infty$ which completements the result of c).

For the first claim of b), we can compute directly

$$\|u(\cdot,t) - u_0\|_{L^2(U)}^2 = \sum_{n=1}^{\infty} |a_n^2| |e^{\lambda_n t} - 1|^2 \leq \varepsilon + \sum_{n=1}^{N} |a_n^2| |e^{\lambda_n t} - 1|^2$$

where $\varepsilon > 0$ is given and $N \in \mathbb{N}$ is chosen large enough. Note that

$$\sum_{n=1}^{N} |a_n^2| |e^{\lambda_n t} - 1|^2 \leq |e^{\lambda_N t} - 1|^2 \sum_{n=1}^{N} |a_n^2| \leq |e^{\lambda_N t} - 1|^2 \sum_{n=1}^{\infty} |a_n^2|$$

goes to zero as $t \to \infty$ which proves the first claim of b).

For the second claim we first use Exercise 1, by which it is enough to prove that

$$\sum_{n=1} |a_n|^2 |\lambda_n| e^{2|\lambda_n| t}$$

is convergent. Note that since the series $\sum_{n=1}^{\infty} |a_n|^2 = \|u_0\|_{L^2(U)}$ is convergent, the sequence $(a_n)_n$ is bounded and so there is some $M > 0$ such that

$$\sum_{n=1} |a_n|^2 |\lambda_n| e^{2|\lambda_n| t} \leq M \sum_{n=1} |\lambda_n| e^{2|\lambda_n| t}.$$

By Exercise 2 there exist constants $C_1, C_2 > 0$ such that

$$C_1 n^{\frac{2}{d}} \leq |\lambda_n| t \leq C_2 n^{\frac{2}{d}}. \tag{1}$$

This gives

$$\sum_{n=1} |\lambda_n| e^{2|\lambda_n| t} \leq \sum_{n=1} C_2 n^{\frac{2}{d}} e^{-2C_1 n^{\frac{2}{d}} t}$$

but the latter is convergent as it is the pointwise product of a polynomial in n and an exponentially decaying function in n as $t > 0$. This proves the claim.

c) For any compact set $K \subset U$ we have

$$\sum_{n=1}^{\infty} |a_n| e^{-|\lambda_n| t} \|f_n\|_{K,\infty} \ll \sum_{n=1}^{\infty} |a_n| e^{-|\lambda_n| t} |\lambda_n|^{\frac{d}{4}+\frac{1}{2}}$$

by Exercise 5. Using the notation from b) and in particular from (1) again we get

$$\sum_{n=1}^{\infty} |a_n| e^{-|\lambda_n| t} |\lambda_n|^{\frac{d}{4}+\frac{1}{2}} \ll \sum_{n=1}^{\infty} n^{\varkappa} e^{-2C_1 n^{\frac{2}{d}} t}$$

for some $\varkappa > 0$. This series is convergent by the same argument as in b). So since $C(K)$ is a Banach space, u is continuous on K. But K was arbitrary and thus $u(\cdot, t)$ is continuous on U.

It remains to check that $\|u(\cdot, t)\|_{K,\infty} \to 0$ as $t \to \infty$. By the above, it sufficient to show that for any $\varkappa > 0$

$$b(t) := \sum_{n=1}^{\infty} n^{\varkappa} e^{-2C_1 n^{\frac{2}{d}} t} \to 0$$

as $t \to \infty$. Note that b is monotonely decreasing and so it suffices to show that for any $\varepsilon > 0$ we can find $t \in \mathbb{R}$ with $b(t) < \varepsilon$. Let $N \in \mathbb{N}$ be such that

$$\sum_{n=N}^{\infty} n^{\varkappa} e^{-2C_1 n^{\frac{2}{d}}} < \varepsilon.$$

Note that for any $\ell \in \mathbb{N}$

$$b(\ell^{\frac{2}{d}}) = \sum_{n=N}^{\infty} n^{\varkappa} e^{-2C_1 n^{\frac{2}{d}} \ell^{\frac{2}{d}}} = \sum_{m \in \mathbb{N},\, \ell | m} \left(\tfrac{m}{\ell}\right)^{\varkappa} e^{2C_1 m^{\frac{2}{d}}} = \ell^{-\varkappa} \sum_{m \in \mathbb{N},\, \ell | m} m^{\varkappa} e^{2C_1 m^{\frac{2}{d}}}$$
$$\leq \ell^{-\varkappa} b(1)$$

which goes to zero as $\ell \to \infty$. This proves the claim.

d) The solution of Exercise 5 consisted in two steps:

1. Estimating the supremum norm on K by the H^k-norm for some k (Exercise 5, Sheet 3) and
2. estimating the H^k-norm by the eigenvalue and the L^2-norm.

For Step 1 we take a quick look at Exercise 5 in Sheet 3 and its solution to see that what one needs to determine is for which $\ell \leq k$ we have an inclusion

$$H^k(\mathbb{T}^d_R) \to C^\ell(\mathbb{T}^d_R).$$

This is answered in Theorem 5.6; we require $k > \ell + \frac{d}{2}$. Therefore, one ought to show that

$$\|\partial_\alpha f\|_{K,\infty} = \|\partial_\alpha(\chi f)\|_{K,\infty} \ll \|\chi f\|_{H^k(U)}$$

for all $\chi \in C_c^\infty(U)$ with $\chi|_K \equiv 1$ and α with $\|\alpha\|_1 \leq \ell$. Then applying the statement in Exercise 5 for ℓ and k with $\ell + \frac{d}{2} < k \leq \ell + \frac{d}{2} + 1$ verbatim we obtain for $\|\alpha\|_1 \leq \ell$

$$\|\partial_\alpha f\|_{K,\infty} \ll \|\chi f\|_{H^k(U)} \ll |\lambda|^{\frac{k}{2}} \|f\|_{L^2(U)} \ll |\lambda|^{\frac{d}{4}+\frac{\ell}{2}+\frac{1}{2}} \|f\|_{L^2(U)}.$$

As was needed in the proof of c) this gives a polynomial rate in the eigenvalue and therefore the partial sums of $u(\cdot, t)$ converge in C^ℓ on K. We have thus shown smoothness in the position.

For smoothness in all parameters we just notice that the time derivatives of the partial sums of $u(\cdot, t)$ look like

$$\sum_{n=1}^{N} a_n \lambda_n^m e^{\lambda_n t} f_n(x)$$

which again changes the coefficient polynomially in the eigenvalue (and thus in n by Exercise 2). This concludes this exercise.