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Solutions for exercise sheet 5

1. Suppose by contradiction that there exists $\varepsilon > 0$ and a set of $|\mu|$-positive measure A with $|g(x)| < 1 - \varepsilon$ for all $x \in A$. We then have for any $f \in C_0(X)$ with $\|f\|_\infty \leq 1$

$$\begin{aligned}|\Lambda(f)| &\leq \int_X |fg| \,\mathrm{d}|\mu| \leq \int_X |g| \,\mathrm{d}|\mu| = \int_A |g| \,\mathrm{d}|\mu| + \int_{X\setminus A} |g| \,\mathrm{d}|\mu| \\ &\leq (1-\varepsilon)|\mu|(A) + |\mu|(X \setminus A) = |\mu|(X) - \varepsilon|\mu|(A).\end{aligned}$$

This shows that $\|\Lambda\|_{\mathrm{op}} \leq |\mu|(X) - \varepsilon|\mu|(A) < |\mu|(X)$ which is a contradiction.

2. **a)** Note that R_α is invertible with inverse given by $R_{-\alpha}$. Thus, $n \in \mathbb{Z} \mapsto R_{n\alpha}$ defines a $\mathbb{Z}$-action on $\mathbb{T}$. The claim is hence contained in Lemma 3.74.

b) Assume that there exists a non-zero eigenfunction $f = \sum_{n\in\mathbb{Z}} a_n\chi_n \in L^2(\mathbb{T})$ with eigenvalue 1. In terms of the orthogonal decomposition into characters this means

$$\begin{aligned}U_\alpha(f) &= \sum_{n\in\mathbb{Z}} a_n U_\alpha(\chi_n) = \sum_{n\in\mathbb{Z}} a_n \chi_n(\alpha)\chi_n \\ &= f = \sum_{n\in\mathbb{Z}} a_n\chi_n\end{aligned}$$

Comparing coefficients we obtain $a_n\chi_n(\alpha) = a_n$ for every $n \in \mathbb{Z}$. Since f is non-zero, there is $n \in \mathbb{Z}$ for which a_n is non-zero. Hence, we have for this n that $\chi_n(\alpha) = 1$. Explicitly, this means $\chi_n(\alpha) = e^{2\pi i n\alpha} = e^0$ and hence $n\alpha \in \mathbb{Z}$. This implies that $\alpha \in \mathbb{Q}$. Note that we implicitly identified $\mathbb{T}$ with $[0,1)$, but any element in the torus has a rational representative if and only if all its representatives are rational.

Conversely, if $\alpha = \frac{p}{q}$ is rational then $q\alpha \in \mathbb{Z}$ which yields $\chi_q(\alpha) = 1$. This shows that χ_q is an eigenfunction of U_α with eigenvalue 1.

c) Above, we already used that $U_\alpha(\chi_n) = \chi_n(\alpha)\chi_n$ or in other words that χ_n is an eigenfunction for eigenvalue $\chi_n(\alpha)$. The characters thus give an orthonormal decomposition of $L^2(\mathbb{T})$ into eigenfunctions.

3. **a)** Since the argument using Theorem 7.54 will appear in b), let us use the Lebesgue decomposition instead. Write $\mu = \mu_1 - \mu_2$ as in the definition of a real-valued signed measure and decompose $\mu_2 = \mu_{2,\text{abs}} + \mu_{2,\text{sing}}$ into finite positive measures where $\mu_{2,\text{abs}} \ll \mu_1$ and $\mu_{2,\text{sing}} \perp \mu_1$. Then

$$\mu = \mu_1 - \mu_2 = (\mu_1 - \mu_{2,\text{abs}}) - \mu_{2,\text{sing}}.$$

Note that since $\mu_{2,\text{abs}}$ is absolutely continuous with respect to μ_1 we may find $h \in L^1_{\mu_1}(X)$ with $\mathrm{d}\mu_{2,\text{abs}} = h \,\mathrm{d}\mu_1$. Then

$$\mathrm{d}\mu_1 - \mathrm{d}\mu_{2,\text{abs}} = (1-h)\,\mathrm{d}\mu_1$$

Let A^+ be the measurable set where $1 - h \geq 0$ and A^- the measurable set where $1 - h < 0$. Then we define

$$\mu_+ := \mu_1|_{A^+} - \mu_{2,\text{abs}}|_{A^+},$$
$$\mu_- := \mu_{2,\text{sing}} - (\mu_1|_{A^-} - \mu_{2,\text{abs}}|_{A^-}).$$

One checks that $\mu = \mu_+ - \mu_-$ and that μ_+ and μ_- are positive measures singular with respect to each other.

Now let $C \subset X$ be measurable such that $\mu_+(X \setminus C) = \mu_-(C) = 0$ and let $A \subset X$ be measurable. Then

$$\mu(A \cap C) = \mu_+(A \cap C) - \mu_-(A \cap C) = \mu_+(A).$$

This shows that

$$\mu_+(A) = \mu(A \cap C) \leq \sup\{\mu(B) : B \subset A \text{ measurable}\}.$$

Conversely, if $B \subset A$ is measurable we have by the argument from above

$$\mu(B) = \mu_+(B) - \mu_-(B) \leq \mu_+(B) = \mu_+(B \cap C) \leq \mu_+(A \cap C) = \mu_+(A)$$

which deduces the first equality. The second is proven analogously or by noting that one can apply the first equality to $-\mu$.

b) Let $\mu = \mu_{\text{real}} + \mathrm{i}\mu_{\text{imag}}$ be a complex signed measure. We apply Theorem 7.54 to the two real signed measures $\mu_{\text{real}}, \mu_{\text{imag}}$. This shows that there exists two measurable functions $g_{\text{real}}, g_{\text{imag}} : X \to \mathbb{R}$ bounded by 1 and two positive finite measures $\mu_{\text{real}}, \mu_{\text{imag}}$with the property that

$$\int_X f \,\mathrm{d}\mu_{\text{real}} = \int_X f g_{\text{real}} \,\mathrm{d}|\mu_{\text{real}}|$$
$$\int_X f \,\mathrm{d}\mu_{\text{imag}} = \int_X f g_{\text{imag}} \,\mathrm{d}|\mu_{\text{imag}}|$$

for all $f \in C_0(X)$. As was established in Exercise 1, we can assume that we have $g_{\text{real}}(x) \in \{\pm 1\}$ for $|\mu_{\text{real}}|$-almost every $x \in X$ and similarly for g_{imag}. Now consider the measure

$$\nu := |\mu_{\text{real}}| + |\mu_{\text{imag}}|$$

By definition, both $|\mu_{\text{real}}|$ and $|\mu_{\text{imag}}|$ are absolutely continuous with respect to ν and so we may find $h_{\text{real}} : X \to \mathbb{R}$ integrable (with respect to ν) and $h_{\text{imag}} : X \to \mathbb{R}$ integrable with

$$h_{\text{real}}\,\mathrm{d}\nu = \mathrm{d}|\mu_{\text{real}}|, \quad h_{\text{imag}}\,\mathrm{d}\nu = \mathrm{d}|\mu_{\text{real}}|$$

Putting everything together, we obtain

$$\int_X f\,\mathrm{d}\mu_{\text{real}} = \int_X f g_{\text{real}} h_{\text{real}}\,\mathrm{d}\nu, \quad \int_X f\,\mathrm{d}\mu_{\text{imag}} = \int_X f g_{\text{imag}} h_{\text{imag}}\,\mathrm{d}\nu.$$

This proves that $\int_X f\,\mathrm{d}\mu = \int_X f g_1\,\mathrm{d}\nu$ where

$$g_1 = g_{\text{real}} h_{\text{real}} + \mathrm{i} g_{\text{imag}} h_{\text{imag}}.$$

It remains to normalize g_1. So let $g_2 = |g_1|$ and $h = \frac{g_1}{|g_1|}$ where we set $h(x) = 0$ if $g_1(x) = 0$. Consider the measure ν' with $\mathrm{d}\nu' = g_2\,\mathrm{d}\nu$. Then

$$\int_X f\,\mathrm{d}\mu = \int_X f g_1\,\mathrm{d}\nu = \int_X f h g_2\,\mathrm{d}\nu = \int_X f h\,\mathrm{d}\nu'$$

which concludes the claim.

4. Note that $L^2_m(G)$ has an orthonormal basis given by functions of the form [1]

$$\psi_{m,n} : (k, x) \in \mathbb{Z} \times \mathbb{T} \mapsto \delta_{m,k}\chi_n(x).$$

a) Let $g = (k_0, x_0) \in G$ be given. Note that

$$\begin{aligned} U_g(\psi_{m,n})(k,x) &= \psi_{m,n}(k + k_0, x + x_0) = \delta_{m,k+k_0}\chi_n(x + x_0) \\ &= \chi_n(x_0)\delta_{m-k_0,k}\chi_n(x) = \chi_n(x_0)\psi_{m-k_0,n}(k,x). \end{aligned}$$

Thus, if $k_0 = 0$ the functions $\psi_{m,n}$ are eigenfunctions.

[1] To see this, apply "Fubini", which shows that any function f can be written as $\sum_{n\in\mathbb{Z}} \delta_n \cdot f(n, \cdot)$ and then decompose the latter using characters. Alternatively, the statement follows from applying the theorem of Stone-Weierstrass in the right formulation.

Now assume that $k_0 \neq 0$ and that U_g has an eigenvector f for eigenvalue λ. Write $f = \sum_{m,n} a_{mn}\psi_{m,n}$ and note that by the above calculation

$$
\begin{aligned}
U_g(f) &= \sum_{m,n\in\mathbb{Z}} a_{m,n}\chi_n(x_0)\psi_{m-k_0,n} = \sum_{m,n\in\mathbb{Z}} a_{m+k_0,n}\chi_n(x_0)\psi_{m,n} \\
&= \sum_{m,n} \lambda a_{mn}\psi_{m,n}.
\end{aligned}
$$

By comparison of coefficients we have $\lambda a_{mn} = a_{m+k_0,n}\chi_n(x_0)$ and in particular[2] $|a_{m,n}| = |a_{m+k_0,n}|$ for any $m, n \in \mathbb{Z}$. But $\sum_{m,n\in\mathbb{Z}} |a_{m,n}|^2 < \infty$. This is impossible: if (m, n) is such that $|a_{m,n}| \neq 0$ we have an infinite ray (infinite since $k_0 \neq 0$)

$$
|a_{m,n}| = |a_{m+k_0,n}| = |a_{m+2k_0,n}| = \dots
$$

of equal, non-zero values appearing in the series $\sum_{m,n\in\mathbb{Z}} |a_{m,n}|^2$. In conclusion, we have proven that U_g for $g = (k_0, x_0)$ has purely continuous spectrum if and only if $k_0 \neq 0$.

b) Let $g \in G$ such that U_g does not have purely continuous spectrum. By a) this means that $g = (0, \alpha)$ for $\alpha \in \mathbb{T}$. But then the functions $\psi_{m,n}$ defined in a) form an orthonormal basis of eigenfunctions. Thus, U_g has purely discrete spectrum if it does not have purely continuous spectrum. In particular, there is no $g \in G$ for which U_g has mixed spectrum.

5. a) Following the hint we first embed $\ell^\infty(\mathbb{N})$ into $\mathscr{L}^\infty(X)$. For this, we partition $X = [0, 1]$ into the intervals

$$
I_n = \Big(\frac{1}{n+1}, \frac{1}{n}\Big]
$$

for $n \in \mathbb{N}$. We then define an embedding $\Phi : \ell^\infty(\mathbb{N}) \to \mathscr{L}^\infty(X)$ via

$$
\Phi((a_n)_n)(x) = a_m
$$

if $x \in I_m$. By definition this embedding is isometric. The Banach limit LIM on $\ell^\infty(\mathbb{N})$ can be thus viewed as a bounded linear functional on $\Phi(\ell^\infty(\mathbb{N}))$ that we can extend to a bounded linear functional ℓ on $\mathscr{L}^\infty(X)$. It remains to show that there is no signed measure representing ℓ. So suppose on the contrary that

$$
\ell(f) = \int_X f \,\mathrm{d}\mu
$$

[2] Since U_g is unitary, any eigenvalue has absolute value 1.

for every $f \in \mathscr{L}^\infty(X)$ and a signed measure μ on X. In particular, we have for any $m \in \mathbb{N}$

$$\mu(I_m) = \int_X \mathbb{1}_{I_m} \,\mathrm{d}\mu = \ell(\mathbb{1}_{I_m}) = \mathrm{LIM}(e_m) = 0.$$

Here, e_m denotes the sequence which is 1 at the m-coordinate and zero otherwise. In particular, $\mu(X) = 0$ by σ-additivity. However, we have

$$\mu(X) = \int_X 1 \,\mathrm{d}\mu = \ell(1) = \mathrm{LIM}(1) = 1$$

which yields a contradiction.

b) As stated in the exercise we have an operator

$$\mathscr{L}^\infty(X) \to \mathcal{M}(X)^* \tag{1}$$

given associating to $f \in \mathscr{L}^\infty(X)$ the map $\mu \mapsto \int_X f \,\mathrm{d}\mu$. By the way this map is defined, it is clear that the isometric embedding $C(X) \to C(X)^* \cong \mathcal{M}(X)$ factors through it. To show that the latter isometric embedding is not onto, it suffices to show that the map in (1) is injective. Indeed, the map $C(X) \to \mathscr{L}^\infty(X)$ is far from being surjective.

It remains to show that (1) is injective. So let $f \in \mathscr{L}^\infty(X)$ and suppose that $\int_X f \,\mathrm{d}\mu = 0$ for any signed measure on X. In particular, we have for any $x \in X$ that

$$f(x) = \int_X f \,\mathrm{d}\delta_x = 0$$

and thus $f = 0$ as claimed.

6. a) Notice first that the support of $D^h f$ is contained in $\mathrm{supp}(f)+[-|h|, |h|]$ and so we may as well assume that it is contained in (the interior of) $K = \mathrm{supp}(f)+[-1, 1]$ by considering h small enough. Note that since f' is continuous and $\mathrm{supp}(f') \subset \mathrm{supp}(f)$ is compact, we may choose[3] for given $\varepsilon > 0$ some $\delta > 0$ such that

$$|f'(x_1) - f'(x_2)| < \varepsilon$$

whenever $|x_1 - x_2| < \delta$ for $x_1, x_2 \in K$.

By the mean value theorem there exists for any $x_1, x_2 \in K$ some y between x_1 and x_2 such that $f(x_1) - f(x_2) = f'(y)(x_1 - x_2)$. In particular, if h is small

[3] For this, one applies uniform continuity of f' on $\mathrm{supp}(f') + I$ for some large compact interval I containing 0 in the interior and then chooses δ small enough according to the choice of I.

enough and $x \in \operatorname{supp}(f) + [-\frac{1}{2}, \frac{1}{2}]$ there is $y \in K$ between $x+h$ and x with $f(x+h) - f(x) = f'(y)h$ and therefore

$$|D^h f(x) - f'(x)| = |f'(y) - f'(x)| < \varepsilon.$$

Outside of the set $\operatorname{supp}(f) + [-\frac{1}{2}, \frac{1}{2}]$ the difference $|D^h f(x) - f'(x)|$ is zero for small enough h. Therefore, for small enough h

$$\|D^h f - f'\|_{L^2(\mathbb{R})}^2 = \int_K |D^h f - f'|^2 \,\mathrm{d}x < \varepsilon^2 m(K).$$

This proves the claim.

b) Since $L^2(\mathbb{R})$ is a Hilbert space, it is in particular reflexive and the weak topology on $L^2(\mathbb{R}) = L^2(\mathbb{R})^*$ coincides with the weak*-topology. Therefore, the Banach-Alaoglu theorem applies as we assumed that $D^h f$ is a bounded collection.

So let $(h^k)_k$ be a sequence for which $D^{h_k} f \to g \in L^2(\mathbb{R})$ in the weak topology, that is,

$$\left\langle D^{h_k} f, \phi \right\rangle \to \langle g, \phi \rangle$$

for any $\phi \in L^2(\mathbb{R})$. We claim that $g = f'$. For this, notice that for any $\phi \in C_c^\infty(\mathbb{R})$

$$\left\langle D^{h_k} f, \phi \right\rangle = -\left\langle f, D^{-h_k} \phi \right\rangle \to -\left\langle f, \phi' \right\rangle$$

where in the equality we used a simple substitution and the latter part follows from a). Thus, $-\langle f, \phi' \rangle = \langle g, \phi \rangle$ for any $\phi \in C_c^\infty(\mathbb{R})$ which shows that the weak derivative f' of f exists and is equal to g.

It remains to show that $\left\langle D^h f, \phi \right\rangle \to \langle f', \phi \rangle$ for all $\phi \in L^2(\mathbb{R})$. Suppose that this is false for some $\phi \in L^2(\mathbb{R})$ or equivalently that there exists some $\varepsilon > 0$ such that

$$H_\varepsilon = \{h \in \mathbb{R} : | \left\langle D^h f, \phi \right\rangle - \langle f', \phi \rangle | \geq \varepsilon\}$$

is unbounded. We may then pick a sequence $(h_k)_k$ of elements in H_ε and apply the Banach-Alaoglu theorem together with the above argument to deduce that $D^{h_k} f \to f'$ in the weak topology. This clearly contradicts the assumption that $h_k \in H_\varepsilon$ for every k.

c) We apply b) to the inner product with $\mathbb{1}_{[x,y]}$ where we assume for simplicity $x < y$. Then we have for $h > 0$

$$\begin{aligned}
\left\langle D^h f, \mathbb{1}_{[x,y]} \right\rangle &= \tfrac{1}{h}\Big(\int_x^y f(t+h) \,\mathrm{d}t - \int_x^y f(t) \,\mathrm{d}t \Big) \\
&= \tfrac{1}{h}\Big(\int_{x+h}^{y+h} f(t) \,\mathrm{d}t - \int_x^y f(t) \,\mathrm{d}t \Big) \\
&= \tfrac{1}{h}\Big(\int_y^{y+h} f(t) \,\mathrm{d}t - \int_x^{x+h} f(t) \,\mathrm{d}t \Big)
\end{aligned}$$

and similarly

$$\begin{aligned}\left\langle D^{-h}f, \mathbb{1}_{[x,y]}\right\rangle &= \tfrac{1}{-h}\Big(\int_x^y f(t-h)\,\mathrm{d}t - \int_x^y f(t)\,\mathrm{d}t\Big)\\ &= \tfrac{1}{-h}\Big(\int_{x-h}^{y-h} f(t)\,\mathrm{d}t - \int_x^y f(t)\,\mathrm{d}t\Big)\\ &= \tfrac{1}{h}\Big(\int_{y-h}^{y} f(t)\,\mathrm{d}t - \int_{x-h}^{x} f(t)\,\mathrm{d}t\Big).\end{aligned}$$

By b) we obtain

$$\begin{aligned}\frac{1}{2h}\int_{y-h}^{y+h} f(t)\,\mathrm{d}t - \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,\mathrm{d}t &= \tfrac{1}{2}(\left\langle D^{h}f, \mathbb{1}_{[x,y]}\right\rangle + \left\langle D^{-h}f, \mathbb{1}_{[x,y]}\right\rangle)\\ &\to \left\langle f', \mathbb{1}_{[x,y]}\right\rangle = \int_x^y f'(t)\,\mathrm{d}t.\end{aligned}$$

On the other hand, the left hand side converges to $f(y) - f(x)$ whenever x and y are Lebesgue points of f. This proves the claim.

d) We apply the statement in c) to $D^h f$ for given h. It shows that for almost every $x \in \mathbb{R}$ we have

$$D^h f(x) = \frac{1}{h}\int_x^{x+h} f'(t)\,\mathrm{d}t = \frac{1}{h}\int_0^h f'(x+t)\,\mathrm{d}t = \int_0^1 f'(x+ht)\,\mathrm{d}t$$

using substitution. Therefore,

$$\begin{aligned}\|D^h f - f'\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}}\Big|\int_0^1 f'(x+th) - f'(x)\,\mathrm{d}t\Big|^2\,\mathrm{d}x\\ &\le \int_{\mathbb{R}}\int_0^1 |f'(x+th) - f'(x)|^2\,\mathrm{d}t\,\mathrm{d}x\\ &= \int_0^1 \|f'(\cdot + th) - f'\|_{L^2(\mathbb{R})}^2\,\mathrm{d}t.\end{aligned}$$

For the latter expression we have that $\|f'(\cdot + h) - f'\|_{L^2(\mathbb{R})}$ converges to zero as $h \to 0$ by arguments that we have seen before (for instance in Exercise 6, Sheet 1). This implies the statement on average and thus the claim.