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Solutions for exercise sheet 6

1. For $n \in \mathbb{Z}$ we compute using the choice of $\mu_{v,w}$ in the hint

$$\int_{\mathbb{T}} \chi_n \, \mathrm{d}\mu_{v,w} = \tfrac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \int_{\mathbb{T}} \chi_n \, \mathrm{d}\mu_{v+\mathrm{i}^k w} = \tfrac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \left\langle U^n(v + \mathrm{i}^k w), v + \mathrm{i}^k w \right\rangle .$$

Now note that for any k

$$\left\langle U^n(v + \mathrm{i}^k w), v + \mathrm{i}^k w \right\rangle = \langle U^n v, v \rangle + \mathrm{i}^k \langle U^n w, v \rangle + \mathrm{i}^{-k} \langle U^n v, w \rangle + \langle U^n w, w \rangle .$$

Examining the sums over k over each of these terms and using that $\sum_{k=0}^{3} \mathrm{i}^k = \sum_{k=0}^{3} \mathrm{i}^{2k} = 0$ we deduce that

$$\int_{\mathbb{T}} \chi_n \, \mathrm{d}\mu_{v,w} = \tfrac{1}{4} \sum_{k=0}^{3} \langle U^n v, w \rangle = \langle U^n v, w \rangle$$

as claimed.

2. Suppose first that $\mu_v = \|v\|^2 \delta_t$ for $t \in \mathbb{T}$. Then $\chi_n = \chi_n(t)$ holds μ_v-almost everywhere and for every n. In particular, $M_{\chi_1}(\mathbb{1}) = \chi_1 \mathbb{1} = \chi_1(t)\mathbb{1}$ holds μ_v-almost everywhere. This means that $\mathbb{1}$ is an eigenfunction of M_{χ_1} with eigenvalue $\chi_1(t)$. Since we have a unitary isomorphism $\mathcal{H}_v \cong L^2_{\mu_v}(\mathbb{T})$ and M_{χ_1} corresponds to U and $\mathbb{1}$ corresponds to v, this shows that v is an eigenfunction with eigenvalue $\chi_1(t)$.

Conversely, if v is an eigenfunction with eigenvalue λ then $\mathbb{1}$ is an eigenfunction of M_{χ_1} with eigenvalue λ. That means

$$M_{\chi_1}^n(\mathbb{1}) = \lambda^n \mathbb{1} = \chi_n$$

which shows that $\chi_n = \lambda^n$ holds μ_v-almost everywhere. This shows that

$$\int_{\mathbb{T}} \chi_n \, \mathrm{d}\mu_v = \lambda^n \int_{\mathbb{T}} \mathrm{d}\mu_v = \lambda^n \left\langle U^0 v, v \right\rangle = \lambda^n \|v\|^2 = \int_{\mathbb{T}} \chi_n \, \mathrm{d}\nu$$

where $\nu = \|v\|^2 \delta_\lambda$. But if μ_v and ν agree on characters, they agree on trigonometric sums and by Stone-Weierstrass on $C(\mathbb{T})$. By density, they thus agree on $L^2(\mathbb{T})$ which contains the characteristic functions and hence $\mu_v = \nu$ as claimed.

3. **a)** It suffices to show that for any $\delta > 0$ the set of atoms x_0 with $\mu(\{x_0\}) > \delta$ is finite (by applying this to $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$). This is however a direct consequence from the fact that μ is a finite measure.

b) For any $N \in \mathbb{N}$ we have

$$\frac{1}{2N+1} \sum_{n=-N}^{N} p_n(\nu) = \frac{1}{2N+1} \sum_{n=-N}^{N} \int_{\mathbb{T}} \chi_n \, \mathrm{d}\nu = \int_{\mathbb{T}} \frac{1}{2N+1} D_N \, \mathrm{d}\nu$$

where $D_N = \sum_{n=-N}^{N} \chi_n$ is the N-th Dirichlet kernel. It follows from Lemma 3.62 that the averages $\frac{1}{2N+1} D_N(x)$ converge to zero for any $x \neq 0$. Since they are bounded by 1 and equal to 1 for $x = 0$, it follows from Lebesgue dominated convergence that

$$\lim_{N \to \infty} \int_{\mathbb{T}} \frac{1}{2N+1} D_N \, \mathrm{d}\nu = \int_{\mathbb{T}} \mathbb{1}_{\{0\}} \, \mathrm{d}\nu = \nu(\{0\}).$$

c) Denote by P the difference map

$$P : (t_1, t_2) \in \mathbb{T}^2 \mapsto t_1 - t_2 \in \mathbb{T}.$$

Recall that the pushforward measure ν is characterized by the property that $\nu(A) = \mu \times \mu(P^{-1}(A))$ for any measurable $A \subset \mathbb{T}$. But $\triangle = P^{-1}(\{0\})$ and so the first claim $\mu \times \mu(\triangle) = \nu(\{0\})$ follows.

For the second claim note that the equation

$$\int_{\mathbb{T}} f \, \mathrm{d}\nu = \int_{\mathbb{T}^2} f \circ P \, \mathrm{d}(\mu \times \mu)$$

is satisfied for any characteristic function $f = \mathbb{1}_A$ where $A \subset \mathbb{T}$ is measurable. Thus, it is also satisfied for any finite linear combination of characteristic functions and by density for any L^1-function.

Applying this to $f = \chi_n$ we obtain

$$\begin{aligned} p_n(\nu) &= \int_{\mathbb{T}} \chi_n \, \mathrm{d}\nu = \int_{\mathbb{T}^2} \chi_n \circ P \, \mathrm{d}(\mu \times \mu) = \int_{\mathbb{T}} \int_{\mathbb{T}} \chi_n(t_1 - t_2) \, \mathrm{d}\mu(t_1) \, \mathrm{d}\mu(t_2) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} \chi_n(t_1) \overline{\chi_n(t_2)} \, \mathrm{d}\mu(t_1) \, \mathrm{d}\mu(t_2) = p_n(\mu) \overline{p_n(\mu)} = |p_n(\mu)|^2 \end{aligned}$$

using Fubini's theorem.

d) Given the first assertion note that Wiener's lemma is immediate by b) and c). To prove the first assertion we use Fubini's theorem again to calculate

$$\begin{aligned} \mu \times \mu(\triangle) &= \int_{\mathbb{T}} \int_{\mathbb{T}} \mathbb{1}_{\triangle}(t_1, t_2) \, \mathrm{d}\mu(t_1) \, \mathrm{d}\mu(t_2) = \int_{\mathbb{T}} \int_{\mathbb{T}} \mathbb{1}_{\{t_2\}}(t_1) \, \mathrm{d}\mu(t_1) \, \mathrm{d}\mu(t_2) \\ &= \int_{\mathbb{T}} \mu(\{t_2\}) \, \mathrm{d}\mu(t_2). \end{aligned}$$

The last integrand is only non-zero on the countable set of atoms and so

$$\int_{\mathbb{T}} \mu(\{t_2\})\,\mathrm{d}\mu(t_2) = \sum_{j=1}^{\infty} \int_{\mathbb{T}} \mathbb{1}_{\{x_j\}}(t_2)\mu(\{t_2\})\,\mathrm{d}\mu(t_2) = \sum_{j=1}^{\infty} \mu(\{x_j\})^2$$

as desired.

4. **a)** By definition the n-th coordinate of $U^k v$ is v_{n-k} which is 1 if $n = k$ and zero otherwise. Let us set for simplicity of notation $e^{(k)} = U^k v$ for $k \in \mathbb{Z}$. The linear hull of the $e^{(k)}$'s is the subspace $c_c(\mathbb{Z})$ of finitely supported sequences and therefore

$$\mathcal{H}_v = \overline{c_c(\mathbb{Z})} = \ell^2(\mathbb{Z})$$

as the first claim states.

To compute the spectral measure we first compute all the inner products of v with its iterates. That is, let $n \in \mathbb{Z}$ be non-zero and compute

$$\langle U^n v, v\rangle = \left\langle e^{(n)}, e^{(0)}\right\rangle = 0 = p_n(\mu_v)$$

Also, if $n = 0$ we have $p_0(\mu_v) = \langle v, v\rangle = 1$. Since a measure is uniquely determined by its Fourier coefficients and the Lebesgue measure has the same Fourier coefficients, we deduce the μ_v is equal to the Lebesgue measure.

b) We note that U_T maps characters to characters of $\mathbb{T}^2$. Indeed, for any $x \in \mathbb{T}^2$

$$U_T(\chi_n)(x) = \chi_n \circ T(x) = \mathrm{e}^{2\pi i n\cdot T(x)} = \mathrm{e}^{2\pi i n\cdot Ax} = \mathrm{e}^{2\pi i A^t n\cdot x} = \chi_{A^t n}(x).$$

This implies that

$$\langle U_T^k \chi_n, \chi_n\rangle = \begin{cases} 1 & \text{if } (A^t)^k n = n \\ 0 & \text{else} \end{cases}.$$

We analyse the cases in which $(A^t)^k n = n$. Certainly, this holds whenever $n = 0$. Otherwise, n is an eigenvector of $(A^t)^k$ and since A is diagonalizable over $\mathbb{R}$, thus also of A^t. Moreover, since n has eigenvalue 1 for $(A^t)^k$ its eigenvalue λ for A^t must be a root of unity. But by assumption the latter is real, so it is either 1 or -1. Since the determinant of A has absolute value 1, the other eigenvalue is equal to $\pm\lambda^{-1}$. If $\det(A) = 1$, this shows that $A = \pm I_2$. If $\det(A) = -1$, we can apply the discussion to A^2 to see that $A^2 = \pm I_2$. In this case, we must have $A^2 = I_2$ as otherwise A would have eigenvalues $\pm\mathrm{i}$ (which contradicts diagonalizability over the reals).

Let us first compute the spectral measures in the non-exceptional case $A^2 = I_2$. Then the above discussion shows that $\langle U_T^k \chi_n, \chi_n \rangle = \delta_{k0}$ when $n \neq 0$ and so by the same argument as in a) we must have $\mu_{\chi_n} = m$ where the latter denotes the Lebesgue measure. For $n = 0$ we note that $\chi_0 = \mathbb{1}$ is in fact a fixed vector under U_T of norm one and so by Exercise 2 we have $\mu_{\chi_0} = \delta_0$.

We treat the exceptional cases in two steps: if $A = I_2$ then every character is a fixed vector and we have $\mu_{\chi_n} = \delta_0$ for every $n \in \mathbb{Z}^2$.

For the remaining exceptional case we assume that $A \neq I_2$ and that $A^2 = I_2$. To start, we also assume that $\det(A) = 1$ (in which case $A = -I_2$). Then we define $v_n^+ = \frac{1}{\sqrt{2}}(\chi_n + \chi_{An})$ and $v_n^- = \frac{1}{\sqrt{2}}(\chi_n - \chi_{An})$ and it follows directly that $U_T(v_n^\pm) = \pm v_n^\pm$. Since $An = -n \neq n$, the two so defined vectors are non-zero, thus orthogonal, and we have

$$\mu_{v_n^+} = \delta_0, \quad \mu_{v_n^-} = \delta_{\frac{1}{2}}.$$

We now compute

$$\begin{aligned}\left\langle U_T^k \chi_n, \chi_n \right\rangle &= \tfrac{1}{2}\left\langle U_T^k(v_n^+ + v_n^-), (v_n^+ + v_n^-)\right\rangle = \tfrac{1}{2}\left\langle U_T^k v_n^+, v_n^+\right\rangle + \tfrac{1}{2}\left\langle U_T^k v_n^-, v_n^-\right\rangle \\ &= \int_{\mathbb{T}} \chi_k \, \mathrm{d}(\tfrac{1}{2}\delta_0 + \tfrac{1}{2}\delta_{\frac{1}{2}}).\end{aligned}$$

Thus, $\mu_{\chi_n} = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\frac{1}{2}}$ and similarly for χ_{-n}.

Let us now treat the slightly techical case where $A^2 = I_2$ and $\det(A) = -1$, that is, A has eigenvalues $1, -1$. The characteristic polynomial $x^2 - 1$ is thus also its minimal polynomial. It follows from uniqueness of the Jordan normal form that A is diagonalizable over $\mathbb{Q}$. By scaling we find A^t-eigenvectors $n_1, n_2 \in \mathbb{Z}^2$ with eigenvalues 1 resp. -1. If n is a scalar multiple of n_1 or n_2 one can apply the above discussion. So let us assume that this is not the case and write $n = a_1 n_1 + a_2 n_2$ for $a_1, a_2 \in \mathbb{Q}$ non-zero. Then $A^t n = a_1 n_2 - a_2 n_2$ is not equal to n and we can apply the same trick as above.

5. **a)** We make the following definition:

A function $p : \mathbb{Z}^d \to \mathbb{C}$ is *positive definite* if for any[1] $c \in c_c(\mathbb{Z}^d)$ we have

$$\sum_{m,n \in \mathbb{Z}^d} c(m)c(n)p(m-n) \geq 0.$$

The analogue of Theorem 9.6 in this context is then the following:

[1] That is, a function $c : \mathbb{Z}^d \to \mathbb{C}$ supported on finitely many points.

Theorem (Herglotz): Let $p : \mathbb{Z}^d \to \mathbb{C}$ be positive definite. Then there exists a uniquely determined measure μ on $\mathbb{T}^d$ for which

$$p(n) = \int_{\mathbb{T}^d} \chi_n \, \mathrm{d}\mu$$

for all $n \in \mathbb{Z}^d$.

To prove this, we proceed exactly as in the proof of Theorem 9.6. Consider the (discrete) hypercube $I_N = [1, N^d] \cap \mathbb{Z}^d$ and for $\vartheta \in \mathbb{T}^d$ the function

$$c : \mathbb{Z}^d \to \mathbb{C}, \quad n \mapsto \begin{cases} \chi_{-n} & \text{if } n \in I_N \\ 0 & \text{else} \end{cases}$$

to see that

$$\begin{aligned} F_N : \vartheta \mapsto & \tfrac{1}{|I_N|} \sum_{m,n \in I_N} \chi_{-m}(\vartheta)\chi_n(\vartheta)p(m-n) \\ &= \tfrac{1}{|I_N|} \sum_{m,n \in I_N} \chi_{-m+n}(\vartheta)p(m-n) \end{aligned}$$

is non-negative. As in the proof of Theorem 9.6 one can consider the measure μ_N with density F_N which then has total mass $p(0)$. Consider a weak*-convergent subsequence $\mu_{N_k} \to \mu$. Then for $\ell \in \mathbb{Z}^d$

$$\begin{aligned} \int_{\mathbb{T}^d} \chi_\ell \, \mathrm{d}\mu &= \lim_{k \to \infty} \int_{\mathbb{T}^d} \chi_\ell(\vartheta) F_{N_k}(\vartheta) \, \mathrm{d}\mu(\vartheta) \\ &= \lim_{k \to \infty} \tfrac{1}{|I_{N_k}|} \sum_{m,n \in I_{N_k}} p(m-n) \int_{\mathbb{T}^d} \chi_{\ell-m+n}(\vartheta) \, \mathrm{d}\mu(\vartheta) \\ &= p(\ell) \lim_{k \to \infty} \tfrac{1}{|I_{N_k}|} \big| \{ n : n, n+\ell \in I_{N_k} \} \big| \end{aligned}$$

Notice that $I_{N_k} - \ell = \prod_{i=1}^d [1 - \ell_i, N_k - \ell_i] \cap \mathbb{Z}^d$ so that the intersection gives

$$|I_{N_k} \cap (I_{N_k} - \ell)| = \prod_{i=1}^d |[1, N_k - |\ell_i|]| = \prod_{i=1}^d (N_k - |\ell_i|).$$

Dividing this by N_k^d and taking the limit, one obtains 1, which concludes that $\int_{\mathbb{T}^d} \chi_\ell \, \mathrm{d}\mu = p(\ell)$. The uniqueness of the measure μ follows exactly as in the proof of Theorem 9.6.

b) Consider a unitary representation π of $\mathbb{Z}^d$ on $\mathcal{H}$ and suppose that $\mathcal{H}$ is cyclic, that is,

$$\mathcal{H} = \mathcal{H}_v = \overline{\langle \pi_n v : n \in \mathbb{Z}^d \rangle}$$

for a generator $v \in \mathcal{H}$. The analogue of Corollary 9.8 is then the following:

There exists a unitary isomorphism $\phi : \mathcal{H} \to L^2(\mathbb{T}^d, \mu_v)$ such that

$$\phi \circ \pi_n(v) = M_{\chi_n} \circ \phi(v)$$

for all $n \in \mathbb{Z}$.

The proof is completely analogous to the proof of Corollary 9.8. Indeed, one considers also the map

$$\phi : \sum_{n\in\mathbb{Z}^d}{}' c_n U^n v \mapsto \sum_{n\in\mathbb{Z}^d}{}' c_n \chi_n$$

which is an isometry on dense subspaces of $\mathcal{H}$ respectively $L^2_{\mu_v}(\mathbb{T}^d)$.

6. **a)** Note that $\mu(T^{-p(n)}A \cap A) > 0$ is equivalent to $\langle U^{p(n)} \mathbb{1}_A, \mathbb{1}_A \rangle > 0$. If we show that

$$\langle U^{p(n)} f, f \rangle_{L^2(X,\mu)} = 0 \ \forall\, n \in \mathbb{N} \Rightarrow f \in (L^2_{\text{rat}})^\perp$$

then this would imply the claim. To see this, take $f = \mathbb{1}_A$ the characteristic function. Then the assumption that we cannot find an n for the polynomial recurrence is equivalent to $\langle U^{p(n)} \mathbb{1}_A, \mathbb{1}_A \rangle_{L^2(X,\mu)} = 0$ for all n and thus $\mu(A) = \langle \mathbb{1}_A, \mathbb{1}_X \rangle = 0$ since $\mathbb{1}_X \in L^2_{\text{rat}}$ is the constant function, being an eigenfunction of eigenvalue 1. But this contradicts our assumption on A, hence there is some n such that $\mu(T^{-p(n)}A \cap A) > 0$.

b) We let ν_f be the spectral measure of f. Then we calculate

$$\left\| \frac{1}{N} \sum_{n=1}^{N} U^{p(n)} f \right\|^2 = \frac{1}{N^2} \sum_{n,k=1}^{N} \langle U^{p(n)} f, U^{p(k)} f \rangle = \frac{1}{N^2} \sum_{n,k=1}^{N} \langle U^{p(n)-p(k)} f, f \rangle$$

$$= \frac{1}{N^2} \sum_{n,k=1}^{N} \int_{\mathbb{T}} \chi\left((p(n) - p(k))\vartheta\right) d\nu_f(\vartheta) = \int_{\mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^{N} \chi(p(n)\vartheta) \right|^2 d\nu_f(\vartheta).$$

c) By Weyl's equidistribution theorem, $p(n)\vartheta$ equidistributes on the torus for ϑ irrational. This means that in the limit $N \to \infty$, the integrand $\frac{1}{N}\sum_{n=1}^{N} \chi(p(n)\vartheta)$ of the last expression vanishes for all $\vartheta \in \mathbb{Q}^c$. Thus if $\mathbb{Q}$ is a null set with respect to ν_f, $\frac{1}{N}\sum_{n=1}^{N} \chi(p(n)\vartheta) \to 0$ almost everywhere with respect to ν_f. It is clearly uniformly bounded, so that by dominated convergence the integral vanishes as well.

d) Since a countable union of null sets is also a null set, we show that for any rational ϕ, $\nu_f(\{\phi\}) = 0$. If $\phi = \frac{b}{a} + \mathbb{Z}$ then the pointmass $\mathbb{1}_{\{\phi\}}$ is a non-zero vector in $L^2(\mathbb{T}, \nu_f)$, and is invariant under $M^a_{\chi_1}$ since

$$M^a_{\chi_1}(\mathbb{1}_{\{\phi\}}) = \chi_a \mathbb{1}_{\{\phi\}} = \chi(a\phi)\mathbb{1}_{\{\phi\}} = 1\mathbb{1}_{\{\phi\}} = \mathbb{1}_{\{\phi\}}.$$

By Corollary 9.8, we deduce that the non-zero vector $g \in \mathcal{H}_f \subset L^2(X,\mu)$ corresponding to $\mathbb{1}_{\{\phi\}}$ is U^a-invariant. But since $f \in (L^2_{\text{rat}})^\perp$ and U leaves this space invariant, also $g \in (L^2_{\text{rat}})^\perp$.

This is a contradiction by the following reasoning. The above shows that the cyclic subspace $\mathcal{H}_g$ generated by g is finite-dimensional and more precisely any element of $\mathcal{H}_g$ is a linear combination of $g, Ug, \ldots, U^{a-1}g$. But since $\mathcal{H}_g$ is U-invariant, it follows from linear algebra that $U|_{\mathcal{H}_g}$ is diagonalizable. Furthermore, U^a is the identity when restricted to $\mathcal{H}_g$ so the eigenvalue of any eigenvector in $\mathcal{H}_g$ is a root of unity. Thus, $\mathcal{H}_g \subset L^2_{\text{rat}}$ and in particular $g \in L^2_{\text{rat}}$. Together with the above, this gives a contradiction.

e) We note that by density we can find some $\tilde{f}$ which is a finite sum of eigenfunctions and which satisfies $\|\tilde{f} - f\| < \varepsilon$. Write $\tilde{f} = \sum'_{\vartheta \in \mathbb{Q}} g_\vartheta$ for eigenfunctions g_ϑ with eigenvalue $\mathrm{e}^{2\pi \mathrm{i}\vartheta}$ where $\vartheta \in \mathbb{Q}$. We have

$$U\tilde{f} = U\sum{}' g_\vartheta = \sum \chi(\vartheta) g_\vartheta.$$

Let d denote the common denominator of the appearing rational numbers in the definition of $\tilde{f}$. Then $U^d\tilde{f} = \tilde{f}$. Since U is unitary, we conclude

$$\|U^{dn}f - f\| \leq \|U^{dn}f - U^{dn}\tilde{f}\| + \|U^{dn}\tilde{f} - f\| = \|f - \tilde{f}\| + \|\tilde{f} - f\| < 2\varepsilon.$$

f) We can finally put everything together: Let f be such that $\langle U^{p(n)}f, f\rangle_{L^2(X,\mu)} = 0$ for all $n \in \mathbb{N}$ and write $f = f_{\text{rat}} + f_{\text{rat}}^\perp$. We need to show that $f_{\text{rat}} = 0$. We write

$$\begin{aligned}
\|f_{\text{rat}}\|^2 = \langle f_{\text{rat}}, f\rangle &= \left\langle f_{\text{rat}} - \frac{1}{N}\sum_{n=1}^{N} U^{p(dn)}f, f\right\rangle + \left\langle \frac{1}{N}\sum_{n=1}^{N} U^{p(dn)}f, f\right\rangle \\
&= \left\langle f_{\text{rat}} - \frac{1}{N}\sum_{n=1}^{N} U^{p(dn)}f, f\right\rangle + \frac{1}{N}\sum_{n=1}^{N} \langle U^{p(dn)}f, f\rangle \\
&= \left\langle f_{\text{rat}} - \frac{1}{N}\sum_{n=1}^{N} U^{p(dn)}f, f\right\rangle
\end{aligned}$$

where N is arbitrary, d is as in e) for f_{rat} and where we used our assumption on the function f. We now turn to estimating the remaining term. By Cauchy-Schwarz we may as well consider

$$\begin{aligned}
\left\|\frac{1}{N}\sum_{n=1}^{N} U^{p(dn)}f - f_{\text{rat}}\right\| &= \left\|\frac{1}{N}\sum_{n=1}^{N} U^{p(dn)}f_{\text{rat}}^\perp + \frac{1}{N}\sum_{n=1}^{N} U^{p(dn)}f_{\text{rat}} - f_{\text{rat}}\right\| \\
&\leq \left\|\frac{1}{N}\sum_{n=1}^{N} U^{p(dn)}f_{\text{rat}}^\perp\right\| + \left\|\frac{1}{N}\sum_{n=1}^{N} U^{p(dn)}f_{\text{rat}} - f_{\text{rat}}\right\|.
\end{aligned}$$

By e) the term on the right is less than ε and by c) and d) the term on the left converges to zero as $N \to \infty$. But N in the estimate for $\|f_{\text{rat}}\|^2$ was arbitrary which proves that $f_{\text{rat}} = 0$ as desired.