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Solutions for exercise sheet 7

1. Suppose by contradiction that μ has two barycenters $x_1, x_2 \in K$. Let $\ell \in X^*$ be such that $\ell(x_1) \neq \ell(x_2)$. On the other hand, we have

$$\ell(x_1) = \int_K \ell \, \mathrm{d}\mu = \ell(x_2)$$

which gives a contradiction.

2. Clearly, (ii) implies (i). For the converse, assume that (i) holds. We define

$$\mathcal{B} = \left\{ \tfrac{n}{2^k} : 0 < n < 2^k \right\}$$

We prove first the following, more elementary statement. The reader may skip this and turn to the full proof in Claim 2 below.

CLAIM 1: If $x = sx_1 + (1-s)x_2$ for $x_1, x_2 \in K$ and $s \in \mathcal{B}$ then $x = x_1 = x_2$.

We prove this by induction on k in $s = \frac{n}{2^k}$. The statement for $k = 1$ is the statement in (i). So suppose that the claim holds for all $s = \frac{n}{2^k}$ where $0 < n < 2^k$ and all $x_1, x_2 \in K$ Suppose that $x = sx_1 + (1-s)x_2$ for $s = \frac{n}{2^{k+1}}$ and $x_1, x_2 \in K$. If n is divisible by 2, we can write $s = \frac{n'}{2^k}$ and are done by the induction hypothesis. So assume that $n \equiv 1 \mod 2$ and write $n = 2n' + 1$. Then

$$s = \frac{n}{2^{k+1}} = \frac{2n'+1}{2^{k+1}} = \frac{n'}{2^k} + \frac{1}{2^{k+1}} =: s' + \frac{1}{2^{k+1}}.$$

Rewriting x as

$$\begin{aligned} x &= s'x_1 + (1-s)x_2 + \tfrac{1}{2^{k+1}}x_1 = s'x_1 + (1-s')\Big(\tfrac{1-s}{1-s'}x_2 + \tfrac{1}{2^{k+1}(1-s')}x_1\Big) \\ &= s'x_1 + (1-s')\Big(\tfrac{2^{k+1}-n}{2^{k+1}-(n-1)}x_2 + \tfrac{1}{2^{k+1}-(n-1)}x_1\Big) \end{aligned}$$

we see that the expression in the big bracket is a convex combination of elements in K and thus also in K. Since the induction hypothesis applies to s' we have that

$$x = x_1 = \tfrac{2^{k+1}-n}{2^{k+1}-(n-1)}x_2 + \tfrac{1}{2^{k+1}-(n-1)}x_1$$

which is equivalent to $x = x_1 = x_2$. This proves the claim.

To prove the statement of the exercise we assume that $x \in K$ does not satisfy the claim of (ii). Thus we can write $x = tx_1 + (1-t)x_2$ where $x_1, x_2 \in K$ are distinct from x and where $t \in (0,1)$. Denoting by d the metric on K we set $\varepsilon = \min(\mathrm{d}(x, x_1), \mathrm{d}(x, x_2)) > 0$. Interchanging x and y we may assume that $t < \frac{1}{2}$. We prove by induction that we can "move" t closer to $\frac{1}{2}$. In the limit one will then attain a contradiction to (i).

CLAIM 2: For every $n \in \mathbb{N}$ there exists some $t_n \in (\frac{1}{2} - 2^{-n}, \frac{1}{2})$ and some $y_n, z_n \in K$ with $\mathrm{d}(x, y_n) \geq \varepsilon$ such that

$$x = t_n y_n + (1 - t_n) z_n.$$

The proof uses the same trick as Claim 1 and proceeds by induction. The base case is the assumption on x. So assume that $x = t_n y_n + (1 - t_n) z_n$ as in the claim. If $\frac{1}{2} - 2^{-n-1} < t_{n+1}$ we are done. Otherwise, we set $t_{n+1} = t_n + 2^{n+1} \in (\frac{1}{2} - 2^{-n-1}, \frac{1}{2})$ and write

$$\begin{aligned} x &= t_{n+1} y_n + (t_n - t_{n+1}) y_n + (1 - t_n) z_n \\ &= t_{n+1} \underbrace{y_n}_{=: y_{n+1}} + (1 - t_{n+1}) \underbrace{\Big(\frac{t_n - t_{n+1}}{1 - t_{n+1}} y_n + \frac{1 - t_n}{1 - t_{n+1}} z_n \Big)}_{=: z_{n+1} \in K}. \end{aligned}$$

This proves the claim.

Using the result of Claim 2 and compactness of K we can pick subsequences $(y_{n_k})_k, (z_{n_k})_k$ such that $y_{n_k} \to y_0$ and $z_{n_k} \to z_0$. In particular,

$$x = t_{n_k} y_{n_k} + (1 - t_{n_k}) z_{n_k} \to \tfrac{1}{2} y_0 + \tfrac{1}{2} z_0$$

where $\mathrm{d}(x, y_0) \geq \varepsilon$. But thus the right hand side is equal to x, which implies by (i) that $y_0 = x$. This is a contradiction.

3. Suppose by contradiction that $f \in K$ is an extremal point. Then certainly $\|f\|_1 = 1$ as otherwise

$$f = \lambda(\tfrac{1}{\lambda} f) + (1 - \lambda) 0$$

for some $\lambda \in (0, 1)$ forms a non-trivial convex combination representing f. The function

$$g : a \in [0, 1] \mapsto \int_0^a |f(t)| \, \mathrm{d}t$$

is continuous. We pick $a \in [0, 1]$ such that $g(a) = \frac{1}{2}$ and write

$$f = f|_{[0,a]} + f|_{[a,1]} = \tfrac{1}{2}(2f|_{[0,a]}) + \tfrac{1}{2}(2f|_{[a,1]}).$$

By construction $2f|_{[0,a]}, 2f|_{[a,1]} \in K$ which is a contradiction to the extremality of f.

4. We follow the instruction in the hint and proceed by induction. The base case is quite immediate: if $K \subset \mathbb{R}$ is convex and compact, it is in particular connected and by compactness it is a compact interval. Any element of a compact interval is a convex combination of its $2 = n + 1$ endpoints.

Assume now that the statement of the exercise holds in dimensions $< n$ and let $K \subset \mathbb{R}^n$ be convex and compact (and non-empty).

CLAIM 1: For any point $x \in \partial K$ there is a closed halfspace $V \subset \mathbb{R}^n$ such that $K \subset V$ and $x \in \partial V$.

To prove the claim using Theorem 8.73 (which is sort of an overkill here). Take a sequence of points $x_k \notin K$ with $x_k \to x$ as $k \to \infty$. By the separation theorem (Theorem 8.73) we may find for given k some $\ell_k \in (\mathbb{R}^n)^*$ and some constant $c_k \in \mathbb{R}$ such that

$$\ell_k(y) \leq c_k < \ell_k(x_k) \tag{1}$$

for all $y \in K$. Dividing with the norm of ℓ_k and adapting c_k accordingly we can assume that $\|\ell_k\|_{\mathrm{op}} = 1$ for all k. In particular, as closed bounded sets in finite-dimensional spaces are compact we can assume after passing to a subsequence that $\ell_k \to \ell \in (\mathbb{R}^n)^*$. The same applies to the sequence of constants c_k so that we can assume $c_k \to c \in \mathbb{R}$ as $k \to \infty$. Taking the limit in (1) we obtain

$$\ell(y) \leq c \leq \ell(x)$$

for all $y \in K$. We therefore set

$$V = \{z \in \mathbb{R}^n : \ell(z) \leq c\}$$

and obtain the claim.

CLAIM 2: Any $x \in \partial K$ can be written as a convex combination of n extremal points of K.

To prove the claim, let V be as in Claim 1 and set $K' = K \cap \partial V = \partial K \cap \partial V$. Then K' is a convex compact subset of the $(n-1)$-dimensional vector space $\mathbb{R}^{n-1} \cong \partial V$. By the induction hypothesis, there are n extremal points $x_1, \ldots, x_n$ of K' such that x is a convex combination of these. It remains to show that any x_i is in fact an extremal point of K. If $x_i = ty + (1-t)z$ for $y, z \in K$ and $t \in (0, 1)$ then since $x \in \partial V$ we must have $y, z \in \partial V$ and hence $y, z \in K'$. Since x_i is extremal in K' we thus have $y = z = x_i$. This proves the claim.

We turn to the statement of the exercise. Let $x \in K$ be arbitrary and let $x_0 \in \partial K$ be an extremal point. Let L be the line through x and x_0. Then $L \cap K$ is compact and convex, hence an interval and we may find another endpoint x' (the given one is

x_0). Then $x' \in \partial K$ and can thus be written as a convex combination of n extremal points of K. Since x is a convex combination of x_0 and x', we can write x as a convex combination of $(n+1)$-points as claimed.

5. We will only consider real-valued sequences below but note that the arguments generalize to complex-valued sequences as the extremal points of the closed unit ball in $\mathbb{C}$ are exactly does lying on the boundary.

a) We claim that the set of extremal points $\operatorname{ext}(K)$ is equal

$$M = \{x \in \ell^\infty(\mathbb{N}) : |x_n| = 1 \text{ for all } n \in \mathbb{N}\}.$$

To prove the claim we show two inclusions. So suppose that there is $x \in \operatorname{ext}(K)$ with $|x_n| < 1$ for some $n \in \mathbb{N}$. Then we can write $x_n = t \cdot 1 + (1-t) \cdot (-1)$ for some $t \in (0, 1)$. This shows that $x = ty + (1-t)z$ where $y \in K$ is defined by

$$y_n = \begin{cases} x_k & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$$

and where z is defined analogously. Thus, $\operatorname{ext}(K) \subset M$.

It remains to show that any point in M is extremal. So let $x \in M$ and write $x = ty + (1-t)z$ for $y, z \in K$ and $t \in (0, 1)$. Then for any $n \in \mathbb{N}$ we have $\pm 1 = x_n = ty_n + (1-t)z_n$ but since $y_n, z_n \in [-1, 1]$ this shows that $x_n = y_n = z_n$. We conclude that $x = y = z$ which shows that $M \subset \operatorname{ext}(K)$ and concludes this part of the exercise.

b) We need to show that $M = \operatorname{ext}(K)$ is closed. Given any sequence $(x_k)_k$ in M with $x_k \to x$ as $k \to \infty$ we know that the sequence of n-th components of the x_k's converges to the n-th component of x which is hence either 1 or -1. This shows that M is closed.

6. **a)** We take several concrete "testing sequences" $c \in c_c(\mathbb{Z})$ as in the definition of positive definiteness. Let $N \in \mathbb{Z}$ be fixed and let $\alpha, \beta \in \mathbb{C}$. Then we choose $c \in c_c(\mathbb{Z})$ with

$$c_n = \begin{cases} \alpha & \text{if } n = 0 \\ \beta & \text{if } n = N \\ 0 & \text{else} \end{cases}.$$

Plugging this into the definition of positive definiteness we obtain that

$$0 \leq |\alpha|^2 + |\beta|^2 + p_N \beta \overline{\alpha} + p_{-N} \alpha \overline{\beta}. \tag{2}$$

Let us first show that this gives $p_{-N} = \overline{p_N}$.

- Setting $\alpha = \beta = 1$ we get that (since implicitly the right hand side of (2) is real) $p_N + p_{-N} \in \mathbb{R}$ or in other words that $\mathrm{Im}(p_{-N}) = -\mathrm{Im}(p_N)$.
- Setting $\alpha = \mathrm{i}$ and $\beta = 1$ we get that $-\mathrm{i}p_N + \mathrm{i}p_{-N} \in \mathbb{R}$. This is equivalent to $p_N - p_{-N} \in \mathrm{i}\mathbb{R}$ or, put differently, $\mathrm{Re}(p_N) = \mathrm{Re}(p_{-N})$.

Together these two bullets indeed show that $p_{-N} = \overline{p_N}$. It remains to show that $|p_N| \leq 1$. For this, we choose $\alpha = p_N$ and $\beta = -1$ in (2) to get $0 \leq 1 - |p_N|^2$ which is what we wanted.

b) By part a), $\mathcal{PD}^1(\mathbb{Z})$ is a subset of the closed unit ball of $\ell^\infty(\mathbb{Z})$ and hence by Tychonoff-Alaoglu we only need to show that $\mathcal{PD}^1(\mathbb{Z})$ is a weak*-closed subset of the closed unit ball. So suppose that we have a sequence $(p^{(k)})_k$ in $\mathcal{PD}^1(\mathbb{Z})$ converging to $p \in \ell^\infty(\mathbb{Z})$. We need to show that p is positive definite. So let $c \in c_c(\mathbb{Z}) \subset \ell^1(\mathbb{Z})$ and note that we have for any $m \in \mathbb{Z}$

$$\sum_{n\in\mathbb{Z}} p^{(k)}_{m-n} c_n = \sum_{n\in\mathbb{Z}} p^{(k)}_{n} c_{m-n} \to \sum_{n\in\mathbb{Z}} p_{m-n} c_n$$

as $k \to \infty$ by assumption. Since c has finite support, this implies

$$\sum_{m,n\in\mathbb{Z}} p^{(k)}_{m-n} c_n c_m \to \sum_{m,n\in\mathbb{Z}} p_{m-n} c_n c_m.$$

Since for any k the sum $\sum_{m,n\in\mathbb{Z}} p^{(k)}_{m-n} c_n c_m$ is non-negative, so is the limit. This proves that $p \in \mathcal{PD}^1(\mathbb{Z})$ as claimed.

To show convexity, let $t \in [0,1]$ and let $p, q \in \mathcal{PD}^1(\mathbb{Z})$. Then we have for any $c \in c_c(\mathbb{Z})$

$$\sum_{m,n\in\mathbb{Z}} (tp + (1-t)q)_{m-n} c_n c_m = t \sum_{m,n\in\mathbb{Z}} p_{m-n} c_n c_m + (1-t) \sum_{m,n\in\mathbb{Z}} q_{m-n} c_n c_m \geq 0$$

which shows that $tp + (1-t)q \in \mathcal{PD}^1(\mathbb{Z})$.

c) Of course, continuity is meant with respect to the weak*-topologies. Let $(\mu_k)_k$ be a sequence of probability measures on $\mathbb{T}$ and assume that it converges in the weak*-topology to some probability measure μ on $\mathbb{T}$. Let $a \in \ell^1(\mathbb{Z})$ and note that

$$\sum_{n\in\mathbb{Z}} p_n(\mu_k) a_n = \int_{\mathbb{T}} \Big(\sum_{n\in\mathbb{Z}} a_n \chi_n \Big) \,\mathrm{d}\mu_k.$$

The sum in the integrand is a continuous function on $\mathbb{T}$ (see also FAI, Sheet 7, Exercise 5) and thus by definition of the weak*topology on $\mathcal{P}(\mathbb{T})$

$$\int_{\mathbb{T}} \Big(\sum_{n\in\mathbb{Z}} a_n \chi_n \Big) \,\mathrm{d}\mu_k \to \int_{\mathbb{T}} \Big(\sum_{n\in\mathbb{Z}} a_n \chi_n \Big) \,\mathrm{d}\mu = \sum_{n\in\mathbb{Z}} p(\mu_k) a_n$$

as $k \to \infty$. This proves the claimed continuity.

To show that the map is affine, let $t \in [0,1]$ and $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{T})$. Then

$$\begin{aligned} p_n(t\mu_1 + (1-t)\mu_2) &= \int_{\mathbb{T}} \chi_n \,\mathrm{d}(t\mu_1 + (1-t)\mu_2) \\ &= t\int_{\mathbb{T}} \chi_n \,\mathrm{d}\mu_1 + (1-t)\int_{\mathbb{T}} \chi_n \,\mathrm{d}\mu_2 \\ &= tp_n(\mu_1) + (1-t)p_n(\mu_2) \end{aligned}$$

as claimed.

d) For the first part: Using Herglotz's theorem and c) we simply need to understand the extremal points in $\mathcal{P}(\mathbb{T})$ and compute their images in $\mathcal{PD}^1(\mathbb{Z})$.

CLAIM The extremal points in $\mathcal{P}(\mathbb{T})$ are the Dirac measures.

Let $\mu \in \mathcal{P}(\mathbb{T})$ be extremal. We consider the monotone function $a \in [0,1) \mapsto \mu([0,a)) \in [0,1]$. If there exists a such that $\mu([0,a)) \in (0,1)$ then the set $U = [0,a)$ yields the non-trivial convex combination

$$\mu = \mu(U)\Big(\tfrac{1}{\mu(U)}\mu|_U\Big) + \mu(\mathbb{T}\setminus U)\Big(\tfrac{1}{\mu(\mathbb{T}\setminus U)}\mu|_{\mathbb{T}\setminus U}\Big)$$

which is impossible. Thus $\mu([0,a)) \in \{0,1\}$. We define

$$a = \sup\{a' : \mu([0,a')) = 0\}.$$

Then for any $b > a$ we have $\mu([0,b)) = 1$ and therefore

$$\mu([0,a]) = \mu\Big(\bigcap_{b>a}[0,b)\Big) = 1.$$

Similarly, $\mu([0,a)) = 0$ and thus $\mu(\{a\}) = 1$. In other words, μ is the Dirac measure at a which proves the claim.

To finish d), we just need to compute the Fourier coefficients of δ_a for $a \in \mathbb{T}$. These are given by the desired expressions.

We now turn to the second part of the exercise. Consider the map

$$F : t_0 \in \mathbb{T} \mapsto (\chi_n(t_0))_n \in \operatorname{ext}(\mathcal{PD}^1(\mathbb{Z}))$$

which is as just proven a bijection. As a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, we just need to explain why F is continuous (for the weak*-topology restricted to $\operatorname{ext}(\mathcal{PD}^1(\mathbb{Z}))$). By c) and Herglotz' theorem, $\operatorname{ext}(\mathcal{PD}^1(\mathbb{Z}))$ is homeomorphic to $\operatorname{ext}(\mathcal{P}(\mathbb{T}))$ (via the map in c)) and thus we just need to check that

$$t_0 \in \mathbb{T} \mapsto \delta_{t_0} \in \mathcal{P}(\mathbb{T})$$

is continuous. Given $f \in C_c(X)$ and $\varepsilon > 0$ we just need to show that there is $\delta > 0$ such that whenever $\mathrm{d}(t, t_0) < \delta$

$$|f(t) - f(t_0)| = \Big| \int f\delta_t - \int f\delta_{t_0} \Big| < \varepsilon$$

This is clear from continuity of f.

e) By d) measures on $\mathrm{ext}(\mathcal{PD}^1(\mathbb{Z}))$ correspond to measures on $\mathbb{T}$. By Choquet's theorem given some $p \in \mathcal{PD}^1(\mathbb{Z})$ there exists a measure μ on $\mathrm{ext}(\mathcal{PD}^1(\mathbb{Z}))$ such that for all $\ell \in \ell^\infty(\mathbb{Z})^*$

$$\ell(p) = \int \ell \,\mathrm{d}\mu.$$

Identifying μ with a measure on $\mathbb{T}$ via d) we rewrite this as

$$\ell(p) = \int \ell((\chi_n(t_0))_n) \,\mathrm{d}\mu(t_0)$$

Plugging in $\ell : (x_n)_n \mapsto x_n$ we get that for every $n \in \mathbb{Z}$

$$p_n = \int_{\mathbb{T}} \chi_n(t_0) \,\mathrm{d}\mu(t_0).$$

This is of course exactly the statement in Herglotz' theorem.