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Solutions for exercise sheet 8

1. As said in the hint, we will show that the multiplication is associative and that the norm satisfies the property required in the definition of Banach algebras.

For the former we let $a_1, a_2, a_3 \in \mathcal{A}$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and compute

$$\begin{aligned}
&(a_1 + \lambda_1 \mathbb{1}) \cdot \big((a_2 + \lambda_2 \mathbb{1}) \cdot (a_3 + \lambda_3 \mathbb{1})\big) \\
&\qquad = (a_1 + \lambda_1 \mathbb{1}) \cdot \big((a_2 a_3 + \lambda_2 a_3 + \lambda_3 a_2) + \lambda_2 \lambda_3 \mathbb{1}\big) \\
&\qquad = \Big(a_1(a_2 a_3 + \lambda_2 a_3 + \lambda_3 a_2) + \lambda_1(a_2 a_3 + \lambda_2 a_3 + \lambda_3 a_2) + \lambda_2 \lambda_3 a_1\Big) \\
&\qquad\qquad + \lambda_1 \lambda_2 \lambda_3 \mathbb{1} \\
&\qquad = \Big(a_1 a_2 a_3 + \lambda_2 a_1 a_3 + \lambda_3 a_1 a_2 + \lambda_1 a_2 a_3 + \lambda_1 \lambda_2 a_3 + \lambda_1 \lambda_3 a_2 + \lambda_2 \lambda_3 a_1\Big) \\
&\qquad\qquad + \lambda_1 \lambda_2 \lambda_3 \mathbb{1}.
\end{aligned}$$

Quite similarly we calculate

$$\begin{aligned}
&\big((a_1 + \lambda_1 \mathbb{1}) \cdot (a_2 + \lambda_2 \mathbb{1})\big) \cdot (a_3 + \lambda_3 \mathbb{1}) \\
&\qquad = \big((a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1) + \lambda_1 \lambda_2 \mathbb{1}\big) \cdot (a_3 + \lambda_3 \mathbb{1}) \\
&\qquad = \Big((a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1) a_3 + \lambda_3(a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1) + \lambda_1 \lambda_2 a_3\Big) \\
&\qquad\qquad + \lambda_1 \lambda_2 \lambda_3 \mathbb{1} \\
&\qquad = \Big(a_1 a_2 a_3 + \lambda_1 a_2 a_3 + \lambda_2 a_1 a_3 + \lambda_3 a_1 a_2 + \lambda_1 \lambda_3 a_2 + \lambda_2 \lambda_3 a_1 + \lambda_1 \lambda_2 a_3\Big) \\
&\qquad\qquad + \lambda_1 \lambda_2 \lambda_3 \mathbb{1} \\
&\qquad = (a_1 + \lambda_1 \mathbb{1}) \cdot \big((a_2 + \lambda_2 \mathbb{1}) \cdot (a_3 + \lambda_3 \mathbb{1})\big)
\end{aligned}$$

as desired.

To check the norm property we let $a_1, a_2 \in \mathcal{A}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then using that $\mathcal{A}$ is a Banach algebra we get

$$\begin{aligned}
\|(a_1 + \lambda_1 \mathbb{1}) \cdot (a_2 + \lambda_2 \mathbb{1})\| &= \|(a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1) + \lambda_1 \lambda_2 \mathbb{1}\| \\
&= \|a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1\| + |\lambda_1 \lambda_2| \\
&\leq \|a_1 a_2\| + |\lambda_1| \|a_2\| + |\lambda_2| \|a_1\| + |\lambda_1 \lambda_2| \\
&\leq \|a_1\| \|a_2\| + |\lambda_1| \|a_2\| + |\lambda_2| \|a_1\| + |\lambda_1 \lambda_2| \\
&= (\|a_1\| + |\lambda_1|)(\|a_2\| + |\lambda_2|)
\end{aligned}$$

as claimed.

We quickly remark that all other properties that need to be shown are quite immediate from definitions. For example, $\mathcal{A}_{\mathbb{1}}$ is a Banach space as a direct product of Banach spaces.

2. **a)** We check all the properties of a Haar measure. At most places we will use that

$$\begin{aligned} \mathbb{R}_{>0} \times \mathrm{SL}_n(\mathbb{R}) &\to \mathrm{GL}_n^+(\mathbb{R}) = \{g \in \mathrm{Mat}_n(\mathbb{R}) : \det(g) > 0\} \\ (\lambda, g) &\mapsto \lambda g \end{aligned} \tag{1}$$

is a homeomorphism with inverse

$$g \in \mathrm{GL}_n(\mathbb{R}) \mapsto \Big(\det(g), \det(g)^{\frac{1}{n}} g \Big).$$

Note that $\mathrm{GL}_n^+(\mathbb{R})$ is an open subset of $\mathrm{Mat}_n(\mathbb{R})$ and thus any non-empty open subset of $\mathrm{GL}_n^+(\mathbb{R})$ has positive Lebesgue measure. The maps above are in fact group isomorphisms as $\mathbb{R}_{>0}$ (scalar multiples of the identity) and $\mathrm{SL}_n(\mathbb{R})$ commute.

With these remarks we first prove the following two properties.

- Let $K \subset \mathrm{SL}_n(\mathbb{R})$ be compact. Then $[0,1] \cdot K \subset \mathrm{Mat}_n(\mathbb{R})$ is compact – it is the image $[0,1] \times K$ under the continuous map $\mathbb{R} \times \mathrm{SL}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R})$ as in (1). In particular, its Lebesgue measure is finite and hence so is $m_G(K)$.
- Let $U \subset \mathrm{SL}_n(\mathbb{R})$ be open. Then $(0,1) \cdot U \subset \mathrm{GL}_n(\mathbb{R})$ is a non-empty open subset of $\mathrm{GL}_n^+(\mathbb{R})$ by (1) and thus has positive Lebesgue-measure. By definition of m_G we have

$$0 < \lambda((0,1) \cdot U) = \lambda([0,1] \cdot U) = m_G(U).$$

It remains to prove left-invariance and right-invariance. For this, observe first that λ is $\mathrm{SL}_n(\mathbb{R})$-invariant in the following sense. If $g \in \mathrm{SL}_n(\mathbb{R})$ then the map $h \in \mathrm{Mat}_n(\mathbb{R}) \mapsto gh$ applies g to every column of h and so by definition of the Lebesgue-measure it leaves λ invariant. The analogous argument applies to $h \in \mathrm{Mat}_n(\mathbb{R}) \mapsto hg$ (applying g to rows). Let $g \in \mathrm{SL}_n(\mathbb{R})$ and let $B \subset \mathrm{SL}_n(\mathbb{R})$ be measurable. Then $[0,1] \cdot B \subset \mathrm{Mat}_n(\mathbb{R})$ is measurable and we have as just argued

$$m_G(gB) = \lambda([0,1] \cdot (gB)) = \lambda(g([0,1] \cdot B)) = \lambda([0,1] \cdot B) = m_G(B).$$

For right-invariance one proceeds analogously.

b) We use the coordinate system given implicitly in the definition of B. It yields in particular, that B is homeomorphic to $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$.

The most interesting property is left-invariance, so let us show that first. If $f \in L^1_{m_B}(B)$ and $g \in B$ has coordinates a, b then

$$\begin{aligned}\int_B f(gh)\,\mathrm{d}m_B(h) &= \int_B f\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right)\frac{1}{x^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_B f\left(\begin{pmatrix} ax & ay+b \\ 0 & 1 \end{pmatrix}\right)\frac{1}{x^2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_B f\left(\begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix}\right)\frac{1}{(\frac{s}{a})^2}\frac{1}{a}\,\mathrm{d}s\frac{1}{a}\,\mathrm{d}t \\ &= \int_B f\left(\begin{pmatrix} s & t \\ 0 & 1 \end{pmatrix}\right)\frac{1}{s^2}\,\mathrm{d}s\,\mathrm{d}t = \int_B f\,\mathrm{d}m_B\end{aligned}$$

where we substituted $s = ax$ and $t = ay + b$. This proves left-invariance.

One can proceed in exactly the same way to show that $\frac{1}{a}\,\mathrm{d}\,\mathrm{d}b$ yields a right-Haar measure on B. Clearly, this measure is not proportional to m_B and thus m_B cannot be a right Haar measure.

To prove the rest, notice that if $K \subset B$ is compact, there is some $a' > 0$ with $|a_k| \geq a'$ for all $k \in K$ where a_k denotes the a-coordinate of $k \in K$. Then

$$m_B(K) \leq \tfrac{1}{(a')^2}\int_K \mathrm{d}a\,\mathrm{d}b < \infty$$

as $K \subset \mathbb{R}^2$ is compact. Notice also that any open subset of K is an open subset of $\mathbb{R}^2$ (in the given coordinate system) and that $a \mapsto \frac{1}{a^2}$ is a positive function. The remaining property of Haar measures follows from this.

3. a) If G is discrete, $\{e\} \subset G$ is a non-empty open set and thus has positive (Haar) measure. This is another way of saying that $\{e\}$ is an atom for the Haar measure.

So suppose that m_G has an atom $\{g\}$ for $g \in G$. Then for any $h \in G$ by left-invariance

$$m_G(\{h\}) = m_G(h.\{e\}) = m_G(\{e\}) = m_G(g^{-1}.\{g\}) = m_G(\{g\}).$$

In other words, any point is an atom and all atoms have the same measure. Now let $U \subset G$ be an open neighborhood of the identity $e \in G$ with compact closure. We claim that U is finite. Indeed, if there are $k_1, \ldots, k_n \in K$ for some $n \in \mathbb{N}$ then

$$n \cdot m_G(\{e\}) = m_G(\{k_1, \ldots, k_n\}) \leq m_G(U) \leq m_G(\overline{U}) < \infty$$

and so $n \leq \frac{m_G(U)}{m_G(\{e\})}$. This shows that U is finite.

Since points are closed, $U \setminus \{e\}$ is closed and this $\{e\} = U \setminus (U \setminus \{e\})$ is open. Since left-multiplication is a homeomorphism, any other point set is open which shows that G is discrete.

b) If G is compact, it has finite (Haar) measure by definition of the Haar measure. So let us assume that $m_G(G) < \infty$ and let $K \subset G$ be a compact neighborhood of the identity.

Let $g_1, \ldots, g_n \in G$ be such that $g_1K, \ldots, g_nK$ is a maximal collection of disjoint translates of K. Note that there is a finite such collection as these translates all have the same positive measure and as G has finite measure. We claim that

$$G = \bigcup_{i=1}^{n} g_i K K^{-1}$$

which shows that G is compact as a finite union of compact sets. If there is some element $g \in G$ which is not contained in g_iKK^{-1} for any i then gK is disjoint to g_iK for every i. Indeed, if $gK \cap g_iK \neq \emptyset$ we can write $gk_1 = g_ik_2$ for $k_1, k_2 \in K$ which then gives $g = g_ik_2k_1^{-1} \in g_iKK^{-1}$.

4. Throughout this exercise we denote for a continuous map $\vartheta : G \to G$ and any measure μ on G the pushforward measure by $\vartheta_*\mu$. It is defined by $\vartheta_*\mu(B) = \mu(\vartheta^{-1}(B))$ for $B \subset G$ measurable. Note that it also satisfies

$$\int_G f(\vartheta(x)) \,\mathrm{d}\mu(x) = \int_G f \,\mathrm{d}(\vartheta_*\mu) \tag{2}$$

for any f nice enough.

a) We show that $\vartheta_* m_G$ is a left-Haar measure on G. The claim then follows from uniqueness of the left-Haar measure up to positive scalars. So let $g \in G$ and let $A \subset G$ be measurable. Then

$$\vartheta_* m_G(gA) = m_G(\vartheta^{-1}(gA)) = m_G(\vartheta^{-1}(g)\vartheta^{-1}(A)) = m_G(\vartheta^{-1}(A)) = \vartheta_* m_G(A)$$

where we used that ϑ is an automorphism. This proves left-invariance.

If $K \subset G$ is a compact set, then $\vartheta^{-1}(K)$ is compact as we assumed that $\vartheta : G \to G$ is a homeomorphism. Thus, $\vartheta_* m_G(K) = m_G(\vartheta^{-1}(K))$ is finite.

If $U \subset G$ is a non-empty open set, then $\vartheta^{-1}(U)$ is also a non-empty open set and $\vartheta_* m_G(U) = m_G(\vartheta^{-1}(U))$ is positive.

b) For any integrable f and $h \in G$ we have by left-invariance

$$\int_G f(gh^{-1})\,\mathrm{d}m_G(g) = \int_G f(hgh^{-1})\,\mathrm{d}m_G(g) = \int_G f(\vartheta_h(g))\,\mathrm{d}m_G(g).$$

Then using (2) we get

$$\begin{aligned}\int_G f(\vartheta_h(g))\,\mathrm{d}m_G(g) &= \operatorname{mod}(\vartheta_h)\int_G f(g)\,\mathrm{d}m_G(g)\\ &= \triangle_G(h)\int_G f(g)\,\mathrm{d}m_G(g)\end{aligned}$$

as was claimed.

Let us now apply this to show continuity of the modular character $\triangle_G$. Let $K \subset G$ be a compact neighborhood of the identity with $K = K^{-1}$. We may also assume that $m_G(K^2) < 1$. Furthermore, let $f \in C_c(G)$ be such that $\int_G f\,\mathrm{d}m_G = 1$ (by multiplying with a scalar) and such that $\operatorname{supp}(f) \subset K$ (by Urysohn's lemma). Given $\varepsilon > 0$ there is a neighborhood U of the identity $e \in G$ such that for any $g_1, g_2 \in G$

$$g_2 = g_1 h^{-1} \text{ for some } h \in U \implies |f(g_2) - f(g_1)| < \varepsilon.$$

By shrinking U we may assume that $U \subset K$. Using this we get for any $h \in U$

$$\left|\int_G f(gh^{-1})\,\mathrm{d}m_G(g) - \int_G f(g)\,\mathrm{d}m_G(g)\right| < \varepsilon.$$

On the other hand, we have by the proven integral inequality

$$\left|\int_G f(gh^{-1})\,\mathrm{d}m_G(g) - \int_G f(g)\,\mathrm{d}m_G(g)\right| = |\triangle_G(h) - 1|.$$

This proves that $\triangle_G$ is continuous at the identity.

This implies continuity of $\triangle_G$. Indeed, it suffices to prove that $\triangle_G$ is continuous at any other points $g_0 \in G$. Given $\varepsilon > 0$ we let $U \subset G$ be an open neighborhood of the identity such that $|\triangle_G(h) - 1| < \varepsilon$ for any $h \in U$. We then consider the open neighborhood $U' = g_0 U$ of g_0 and observe that for any $g = g_0 h \in U'$

$$|\triangle_G(g) - \triangle_G(g_0)| = |\triangle_G(g_0)||\triangle_G(h) - 1| < \varepsilon|\triangle_G(g_0)|$$

as claimed.

c)
- If G is compact, then $A = \triangle_G(G) \subset \mathbb{R}_{>0}$ is a compact subgroup of $\mathbb{R}_{>0}$. For any $a \in A$ we then have that the sequence $(a^k)_{k\in\mathbb{N}}$ has a convergent subsequence and is in particular bounded. Thus, $a = 1$. This $\triangle_G \equiv 1$ and G is unimodular.

- If G is abelian, then for any $A \subset G$ measurable and any $h \in G$ we have

$$\triangle_G(h)m_G(A) = m_G(h^{-1}Ah) = m_G(A)$$

which implies the claim.

- Suppose that G is perfect. For $g_1, g_2 \in G$ we note that

$$\triangle_G([g_1, g_2]) = \triangle_G(g_1 g_2 g_1^{-1} g_2^{-1}) = \triangle_G(g_1)\triangle_G(g_2)\triangle_G(g_1)^{-1}\triangle_G(g_2)^{-1} = 1$$

as $\mathbb{R}_{>0}$ is abelian. This shows that $[g_1, g_2] \in \ker(\triangle_G)$. By definition, $[G, G]$ is the smallest subgroup containing all commutators and hence $G = [G, G] \subset \ker(\triangle_G)$ which proves the claim.

5. **a)** We compute directly for any $\nu \in \mathcal{M}(G)$ and $f \in \mathcal{L}^\infty(G)$

$$\int_G f(g)\,\mathrm{d}(\nu * \delta_{\{e\}}) = \int_G \int_G f(g_1 g_2)\,\mathrm{d}\nu(g_1)\,\mathrm{d}\delta_e(g_2) = \int_G f\,\mathrm{d}\nu$$

which shows that $\nu * \delta_{\{e\}} = \nu$. The argument for $\delta_{\{e\}} * \nu = \nu$ is analogous.

b) CLAIM: The unit of $L^1_{m_G}(G)$, if it exists, needs to be equal to the unit $\delta_{\{e\}}$ in $\mathcal{M}(G)$.

To prove this, suppose that $\mathbb{1}$ is a unit of $L^1_{m_G}(G)$. Then for any $g \in G$ and any measurable $B \subset G$

$$\mathbb{1} * \mathbb{1}_B(g) = \int_G \mathbb{1}(h)\mathbb{1}_B(h^{-1}g)\,\mathrm{d}m_G(h) = \int_{gB^{-1}} \mathbb{1}(h)\,\mathrm{d}m_G(h).$$

On the other hand, $\mathbb{1} * \mathbb{1}_B(g) = \mathbb{1}_B(g)$ which is one if and only if $g \in B$ (i.e. if and only if $e \in gB^{-1}$). By substituting we see that for any measurable subset $B \subset G$

$$\int_B \mathbb{1}(h)\,\mathrm{d}m_G(h) = \delta_{\{e\}}(B)$$

which proves the claim.

This means that $L^1_{m_G}(G)$ is unital if and only if there exists $f \in L^1_{m_G}(G)$ such that $\mathrm{d}\delta_{\{e\}} = f\,\mathrm{d}m_G$.

If this holds, then $\{e\}$ must have positive measure as f is certainly non-zero. By 3a) this implies that G is discrete. Conversely, if G is discrete, $\{e\}$ is a non-empty open set and letting $f = \delta_e$ we obtain that $\mathrm{d}\delta_{\{e\}} = f\,\mathrm{d}m_G$ as desired.

c) We let $\mu = m_G^{(r)}$ be the measure defined by

$$\int_G f(g)\,\mathrm{d}\mu(g) = \int_G f(g)\triangle_G(g)^{-1}\,\mathrm{d}m_G(g).$$

If $R_h : G \to G,\ g \mapsto gh$ then for $f \in C_c(G)$

$$\begin{aligned}\int_G f(g)\,\mathrm{d}(R_h)_*\mu(g) &= \int_G f(gh)\,\mathrm{d}\mu(g) = \int_G f(gh)\triangle_G(g)^{-1}\,\mathrm{d}m_G(g)\\ &= \triangle_G(h)\int_G f(gh)\triangle_G(gh)^{-1}\,\mathrm{d}m_G(g).\end{aligned}$$

Applying Exercise 4b) to the function $f \cdot \triangle_G^{-1}$ we get

$$\int_G f(g)\,\mathrm{d}(R_h)_*\mu(g) = \int_G f(g)\triangle_G(g)^{-1}\,\mathrm{d}m_G = \int_G f\,\mathrm{d}\mu$$

and so $(R_h)_*\mu = \mu$. In other words, μ is a right Haar measure and so is the measure $B \mapsto m_G(B^{-1})$. Thus, there is $\alpha > 0$ such that

$$\mu(B) = \alpha\, m_G(B^{-1})$$

for all measurable $B \subset G$. For this, we let $\varepsilon > 0$ and (by continuity of the modular character) let $V \subset G$ be a compact neighborhood with $|\triangle_G(h)^{-1} - 1| < \varepsilon$ for all $h \in V$. As before we can assume that $V^{-1} = V$. Then $\alpha m_G(V^{-1}) = \alpha m_G(V)$ and

$$\begin{aligned}|\mu(V) - m_G(V)| &= \Big|\int_G \mathbb{1}_V(g)\triangle_G(g)^{-1} - \mathbb{1}_V(g)\,\mathrm{d}m_G(g)\Big|\\ &\le \int_V |\triangle_G(g)^{-1} - 1|\,\mathrm{d}m_G(g) < \varepsilon m_G(V).\end{aligned}$$

But therefore

$$|\alpha - 1| m_G(V) = |\mu(V) - m_G(V)| < \varepsilon m_G(V)$$

and dividing with $m_G(V)$ gives the remaining claim.

d) Note first that the formulas in c) together can be reformulated as

$$\int_G f(g^{-1})\,\mathrm{d}m_G = \int_G f(g)\triangle_G(g)^{-1}\,\mathrm{d}m_G$$

and replacing f with $g \mapsto f(g^{-1})$

$$\int_G f(g)\,\mathrm{d}m_G = \int_G f(g^{-1})\triangle_G(g)^{-1}\,\mathrm{d}m_G.$$

Applying this to $|f|$ for some $f \in L^1_{m_G}(G)$ shows that $f^* \in L^1_{m_G}(G)$ and $\|f\|_1 = \|f^*\|_1$. Verifying $(f^*)^* = f$ is a direct calculation.

- Let $f \in L^1_{m_G}(G)$ and let μ_f as on the sheet. Then for ψ measurable and bounded

$$\begin{aligned}\int_G \psi \, \mathrm{d}(\mu_f)^* &= \int_G \psi(g^{-1}) \overline{f(g)^{-1}} \, \mathrm{d}m_G(g) \\ &= \int_G \psi(g) \overline{f(g)} \triangle_G(g)^{-1} \, \mathrm{d}m_G(g) \\ &= \int_G \psi(g) f^*(g) \, \mathrm{d}m_G(g)\end{aligned}$$

  which proves this claim.
- This can be proven by direct calculation. We note however that the first bullet implies that the statement follows from the analogous proven statement for $\mathcal{M}(G)$ proven in the lecture.

6. The claim is the following:

CLAIM: *The spectrum of M_g is the essential range of g.*

Here, the essential range consists of all $\lambda \in \mathbb{C}$ with the property that $\mu(g^{-1}(U)) > 0$ for any neighborhood U of λ. Denote the essential range by $A \subset \mathbb{C}$. We prove two inclusions.

Suppose that $\lambda \notin A$. Then there exists an neighborhood of U of λ so that $\mu(g^{-1}(U)) = 0$. In other words, there is $r > 0$ such that $|g(x) - \lambda| \geq r$ for μ-almost all $x \in X$. Then $x \mapsto (g(x) - \lambda)^{-1}$ is measurable and bounded (up to a nullset). Notice that

$$M_{(g-\lambda)^{-1}}(M_g - \lambda \mathbb{1}) = \mathbb{1}.$$

This proves that $\lambda \notin \sigma(M_g)$.

Conversely, let $\lambda \in A$ and let $\varepsilon > 0$. Then $\mu(g^{-1}(B_\varepsilon(\lambda))) > 0$. In other words, $U_\varepsilon = g^{-1}(B_\varepsilon(\lambda))$ has positive measure and for any $x \in U_\varepsilon$ we have $|g(x) - \lambda| < \varepsilon$. For simplicity we set $h = g - \lambda$ so that $M_g - \lambda \mathbb{1} = M_h$. Let $f_\varepsilon = \frac{1}{\sqrt{\mu(U_\varepsilon)}} \mathbb{1}_{U_\varepsilon}$. Then

$$|M_h f_\varepsilon(x)| = \frac{1}{\sqrt{\mu(U_\varepsilon)}} |h(x)||\mathbb{1}_{U_\varepsilon}(x)| < \varepsilon |f_\varepsilon(x)|$$

and so $\|M_h f_\varepsilon\| \leq \varepsilon \|f\|_\varepsilon = \varepsilon$. This shows that $M_h f_\varepsilon \to 0$ as $\varepsilon \to 0$. However, f_ε does not converge to zero as ε since $\|f_\varepsilon\| = 1$. We have proven that M_h cannot have a continuous inverse and therefore, $\lambda \in \sigma(M_g)$ as $h = g - \lambda$.