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Solutions for exercise sheet 9

1. To prove self-adjointness we apply the involution to the equation $a^* = \mathbb{1}a^* = a^*\mathbb{1}$ for any $a \in \mathcal{A}$ to get

$$a = a\mathbb{1}^* = \mathbb{1}^*a.$$

This proves that $\mathbb{1}^*$ is a unit and by uniqueness $\mathbb{1}^* = \mathbb{1}$.

Using the C^*-property of the norm we now deduce that

$$\|\mathbb{1}\| = \|\mathbb{1}^2\| = \|\mathbb{1}^*\mathbb{1}\| = \|\mathbb{1}\|^2.$$

Since $\mathbb{1}$ is invertible, it is non-zero and so is its norm. Thus, $\|\mathbb{1}\| = 1$ follows from the equation by division with $\|\mathbb{1}\|$.

2. See the corollary.

3. **a)** Certainly, $\mathcal{A}$ with these operations is an algebra. As a product, it is (with either norm) a Banach space by FA I. We need to check that either of the norms satisfies the property required of Banach algebras. Let $a = (a_1, a_2)$ and $b = (b_1, b_2) \in \mathcal{A}$ and compute

$$\begin{aligned}\|ab\|_1 = \|(a_1b_1, a_2b_2)\|_1 &= \|a_1b_1\| + \|a_2b_2\| \leq \|a_1\|\|b_1\| + \|a_2\|\|b_2\| \\ &\leq (\|a_1\| + \|a_2\|)(\|b_1\| + \|b_2\|) = \|a\|_1\|b\|_1\end{aligned}$$

which shows the claim for $\|\cdot\|_1$. For $\|\cdot\|_\infty$ we note that

$$\|a_1b_1\| \leq \|a_1\|\|b_1\| \leq \|a\|_\infty\|b\|_\infty$$

and similarly for $\|a_2b_2\|$.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital and $\mathbb{1}_1$ resp. $\mathbb{1}_2$ denote the respective identity elements, then $\mathbb{1} = (\mathbb{1}_1, \mathbb{1}_2)$ is an identity element of $\mathcal{A}$. If $\mathcal{A}$ is unital and $\mathbb{1} = (a_1, a_2)$ is a unit then for any $x \in \mathcal{A}_1$

$$(x, 0) = (x, 0)(a_1, a_2) = (a_1, a_2)(x, 0) = (xa_1, a_2) = (a_1x, a_2)$$

which proves that $x = xa_1 = a_2x$ and hence $\mathcal{A}_1$ (and similarly $\mathcal{A}_2$) is unital.

Since the multiplication is defined componentwise, the claim about commutativity follows immediately.

b) Assuming that $\mathcal{A}$ is unital, we have by Theorem 11.6 that the limit

$$r = \lim_{n\to\infty} \sqrt[n]{\|a^n\|_1} = \lim_{n\to\infty} \sqrt[n]{\|a_1^n\| + \|a_2^n\|}$$

is the spectral radius of a. Note that this limit does not depend on the choice of norm as in a). We set r_1 and r_2 to be the spectral radii of a_1 respectively a_2. Without loss of generality we may assume that $r_1 \geq r_2$. We need to show that $r = r_1$. Quite easily, one attains

$$r = \lim_{k\to\infty} \sqrt[n_k]{\|a_1^{n_k}\| + \|a_2^{n_k}\|} \geq \lim_{k\to\infty} \sqrt[n_k]{\|a_1^{n_k}\|} = r_1$$

For the converse inequality we let $\varepsilon > 0$ and note that for any n large enough

$$\|a_1^n\| + \|a_2^n\| \leq (r_1 + \varepsilon)^n + (r_2 + \varepsilon)^n \leq 2(r_1 + \varepsilon)^n.$$

Taking the n-th root and the limit as $n \to \infty$ we get $r \leq r_1 + \varepsilon$ and since ε was arbitrary this proves $r \leq r_1$.

c) As alluded to in the hint we think in terms of maximal ideals (which is not necessary, but helpful). In fact, we have the following:

CLAIM: Any ideal in $\mathcal{A}$ is of the form $I_1 \times I_2$ where I_1 and I_2 are ideals in $\mathcal{A}_1$ respectively $\mathcal{A}_2$.

Using the claim for now, we see that any maximal ideal is of the form $I_1 \times \mathcal{A}_2$ where I_1 is a maximal ideal of $\mathcal{A}_1$ or of the form $\mathcal{A}_1 \times I_2$ where I_2 is a maximal ideal of $\mathcal{A}_2$.

Now let $\chi : \mathcal{A} \to \mathbb{C}$ be a non-trivial character. Then $\ker(\chi)$ is a maximal ideal as $\mathcal{A}/\ker(\chi) = \mathbb{C}$. So without loss of generality there is a maximal ideal I_1 in $\mathcal{A}_1$ such that $\ker(\chi) = I_1 \times \mathcal{A}_2$. The maximal ideal I_1 yields a character $\chi_1 : \mathcal{A}_1 \to \mathcal{A}_1/I_1 = \mathbb{C}$ via the quotient map. Then for any $a = (a_1, a_2) \in \mathcal{A}$ we have

$$\chi(a) = \chi(a_1, a_2) = \chi(a_1, 0) = \chi_1(a_1)$$

as claimed.

To prove the claim, we let I be any ideal in $\mathcal{A}$ and let I_1 (resp. I_2) be the projection of I to $\mathcal{A}_1$ (resp. $\mathcal{A}_2$). We need to show that $I = I_1 \times I_2$. If $(a, b) \in I$ then $a \in I_1$ and $b \in I_2$ by definition and so $I \subset I_1 \times I_2$. If $(a, b) \in I$ then $(a, 0) = (\mathbb{1}, 0)(a, b) \in I$ and so $I_1 \times \{0\} \subset I$. Similarly, $\{0\} \times I_2 \subset I$ and by additivity we obtain for any $a \in I_1$ and $b \in I_2$

$$(a, b) = (a, 0) + (0, b) \in I.$$

This proves the claim.

d) Recall that

$$\sigma(a) = \{\chi(a) : \chi : \mathcal{A} \to \mathbb{C} \text{ non-trivial}\}$$

and similarly for a_1, a_2. By part c) the characters fall into two types and so either $\chi(a) = \chi_1(a_1)$ for some χ_1 or $\chi(a) = \chi_1(a_2)$. This shows that

$$\sigma(a) \subset \sigma(a_1) \cup \sigma(a_2).$$

Conversely, any character on $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) yields a character on $\mathcal{A}$ and thus equality holds.

4. **a)** So suppose that J is an ideal strictly containing $\{f : f(x) = 0\}$. Then there is $g \in J$ such that $g(x) \neq 0$. After replacing g with $\frac{1}{g(x)} g \in J$ we may assume that $g(x) = 1$. Then

$$1 = g + (1 - g) \in J$$

as $1 - g$ vanishes at x. This proves that $J = C(X)$.

b) Let J be an ideal which is not contained in any ideal of the form as in a). Then for any $x \in X$ there is a function $f_x \in J$ with $f_x(x) \neq 0$. Since J is an ideal, $|f_x|^2 = \overline{f_x} f_x \in J$. We set $V_x = \{y \in X : f_x(y) \neq 0\}$. By compactness we may find $x_1, \ldots, x_n \in X$ such that $V_{x_1} \cup \ldots \cup V_{x_n} = X$. In other words, for any $x \in X$ there is x_i with $f_{x_i}(x) \neq 0$. Therefore, the function $g = |f_{x_1}|^2 + \ldots + |f_{x_n}|^2$ is nowhere vanishing and thus a unit. However,

$$g = |f_{x_1}|^2 + \ldots + |f_{x_n}|^2 \in J$$

which proves that $J = C(X)$.

c) Let $\chi : \mathcal{A} = C(X) \to \mathbb{C}$ be non-trival character and set $J = \ker(\chi)$, which is a maximal ideal. As already used earlier, χ factors through

$$\mathcal{A} \to \mathcal{A}/J \to \mathbb{C}$$

where the former map is the quotient map and the latter map is unique. Thus, χ is uniquely determined by J. From parts a) and b) we know that $J = \{f : f(x) = 0\}$ for some $x \in X$. If $\chi_x : f \in C(X) \mapsto f(x)$ denotes the character attached to the maximal ideal $\{f : f(x) = 0\}$ this shows that $\chi = \chi_x$.

It remains to show that $X \cong \sigma(C(X))$. For this, we consider the map

$$x \in X \mapsto \chi_x \in \sigma(C(X)). \tag{1}$$

By what we just proved, the map is surjective. It is also injective: if $x_1 \neq x_2$ are two distinct point, there is $f \in C(X)$ with $f(x_1) = 1$ and $f(x_2) = 0$ and so $\chi_{x_1}(f) = 1$ and $\chi_{x_2}(f) = 0$. This shows $\chi_{x_1} \neq \chi_{x_2}$ and thus the map in (1) is bijective.

As any continuous bijective map from a compact space to a Hausdorff space has a continuous inverse, it remains to show that (1) is continuous. Recall that the topology on $\sigma(C(X))$ is the weak*-topology. In other words, we need to show that for any $\varepsilon > 0$, $x \in X$ and $f_1, \ldots, f_n \in C(X)$ there is $\delta > 0$ with the property that

$$|f_i(y) - f_i(x)| = |\chi_y(f_i) - \chi_x(f_i)| < \varepsilon$$

for all $y \in X$ with $\mathrm{d}(x, y) < \delta$ and $i \in \{1, \ldots, n\}$. However, we may let $\delta_i > 0$ be as in the definition of continuity of f_i at x given ε and then let $\delta = \min\{\delta_1, \ldots, \delta_n\}$. This is a δ with the desired property. Thus, the map in (1) is continuous and hence a homeomorphism.

5. **a)** We proceed as in the hint. So let $E \subset \mathbb{N}$ and let $\chi : \mathcal{A} = \ell^\infty(\mathbb{N}) \to \mathbb{C}$ be a non-trivial character. Then $\mathbb{1}_E^2 = \mathbb{1}_E$ which yields

$$\chi(\mathbb{1}_E)^2 = \chi(\mathbb{1}_E^2) = \chi(\mathbb{1}_E)$$

and hence that $\chi(\mathbb{1}_E) \in \{0, 1\}$.

Now let $a \in \ell^\infty(\mathbb{N})$ and $\varepsilon > 0$ and define

$$E_a = \{n : |\chi(a) - a_n| \geq \varepsilon\}.$$

We claim that $\chi(\mathbb{1}_{E_a}) = 0$. For this, define a sequence b by setting $b_n = 0$ if $n \notin E_a$ and $b_n = \frac{1}{\chi(a) - a_n}$ for $n \in E_a$. Thus, $|b_n| \leq \frac{1}{\varepsilon}$ for $n \in E_a$ which shows that $b \in \ell^\infty(\mathbb{N})$. We now observe that

$$\chi\big(b(\chi(a) - a)\big) = \chi(b)\chi(\chi(a) - a) = \chi(b)(\chi(a) - \chi(a)) = 0$$

as $\chi(\mathbb{1}) = 1$. On the other hand, $b(\chi(a) - a) = \mathbb{1}_{E_a}$ which shows the claim.

We now proceed to showing that $\mathbb{N} \subset \beta\mathbb{N}$ is in fact dense. Recall for this that the topology on $\beta\mathbb{N}$ is the weak*-topology and so we let $\varepsilon > 0$, $a^{(1)}, \ldots, a^{(k)} \in \ell^\infty(\mathbb{N})$ and χ be given. We need to find $n \in \mathbb{N}$ such that

$$|\chi(a^{(i)}) - a_n^{(i)}| \geq \varepsilon$$

for all $i \in \{1, \ldots, l\}$. This amounts to showing that there is some $n \in \mathbb{N}$ in the complement of $E := E_{a^{(1)}} \cup \ldots \cup E_{a^{(k)}}$. However,

$$\chi(\mathbb{1}_E) \leq \chi(\mathbb{1}_{E_{a^{(1)}}} + \ldots + \mathbb{1}_{E_{a^{(k)}}}) = \chi(\mathbb{1}_{E_{a^{(1)}}}) + \ldots + \chi(\mathbb{1}_{E_{a^{(k)}}}) = 0$$

Thus, $\chi(\mathbb{1}_E) = 0$. This shows that $\chi(\mathbb{1}_{\mathbb{N}\setminus E}) = 1$ as $\mathbb{1}_E + \chi_{\mathbb{N}\setminus E} = 1$ and hence $\mathbb{N} \setminus E$ is non-empty.

b) For any $a \in \ell^\infty(\mathbb{N})$ we may define $\hat{a} : \chi \in \mathcal{A} \to \chi(a)$. By definition of the weak*-topology $\hat{a} \in C(\beta\mathbb{N})$. We thus have a natural map

$$\ell^\infty(\mathbb{N}) \ni a \mapsto \hat{a} \in C(\beta\mathbb{N}). \tag{2}$$

It is straightforward to check that this is in fact an injective algebra homomorphism. In particular, the image of the map in (2) is a subalgebra $\mathcal{A}'$. We show that the subalgebra $\mathcal{A}'$ is dense by using the Stone-Weierstrass theorem.

- It contains the constant functions as $\hat{\mathbb{1}}(\chi) = \chi(\mathbb{1}) = 1$ for any $\chi \neq 0$.
- It separates points: if $\chi_1 \neq \chi_2$ are two distinct points in $\beta\mathbb{N}$ there is $a \in \ell^\infty(\mathbb{N})$ such that $\chi_1(a) \neq \chi_2(a)$ and so

$$\hat{a}(\chi_1) = \chi_1(a) \neq \chi_2(a) = \hat{a}(\chi_2).$$

- It is closed under complex conjugation: For this, claim that for any $a \in \ell^\infty(\mathbb{N})$ with real values and any χ the value $\chi(a)$ is real. This implies that $\mathcal{A}'$ is closed under complex conjugation as one may decompose any sequence into real and imaginary part.
  To prove the claim, we may show by Theorem 11.23 that $\sigma(a) \subset \mathbb{R}$ To this end, we let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and consider $b = a - \lambda$. Since λ is not real, $b_n \neq 0$ for every n. Also,

$$\begin{aligned}\frac{1}{|b_n|} &= \frac{|a_n - \lambda|}{|a_n - \lambda|^2} = \frac{|a_n - \lambda|}{|a_n - \operatorname{Re}(\lambda)|^2 + |\operatorname{Im}(\lambda)|^2} \\ &\leq \frac{\|a\| + |\lambda|}{|\operatorname{Im}(\lambda)|^2}.\end{aligned}$$

  This shows that b is invertible as claimed.

We now to proving that the map in (2) is surjective. For this, it suffices by the above application of the Stone-Weierstrass theorem to show that the image is complete. We prove that (2) is an isometry, which then implies the claim of the exercise.

Notice that for $a \in \ell^\infty(\mathbb{N})$

$$\|\hat{a}\|_\infty = \sup_\chi |\hat{a}(\chi)| = \sup_\chi |\chi(a)| = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

In words, the sup-norm of $\hat{a}$ is the spectral radius of a. By Theorem 11.6, it is equal to $\lim_{n\to\infty} \sqrt[n]{\|a^n\|}$. Notice however that $\|a^n\| = \|a\|^n$ and so the spectral radius is equal to $\|a\|$. This proves that (2) is an isometry and concludes part b).

c) Assume by contradiction that $\beta\mathbb{N}$ is metrizable. Then Lemma 2.46 implies that $C(\beta\mathbb{N})$ is separable. This contradicts b) as $\ell^\infty(\mathbb{N})$ is not separable.

6. **a)** This is the analogue of Exercise 3c) and d). Set $\mathcal{A} = \bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$ We begin with the first claim of the exercise and use the hint for this. For any χ and any $a \in \mathcal{A}$ we have

$$\chi(a) = \sum_{n=1}^{\infty} \chi(ae_n).$$

For this, one considers the partial sums $\sum_{n=1}^{N} ae_n$ and note that they converge to $a \in \mathcal{A}$ by definition of the direct sum. By linearity, $\sum_{n=1}^{N} \chi(ae_n)$ converges to $\chi(a)$ but also to $\sum_{n=1}^{\infty} \chi(ae_n)$.

Notice that for $m \neq n$ we have $\chi(ae_n)\chi(ae_m) = 0$ as $ae_n ae_m = 0$. Thus, one of $\chi(ae_n)$ and $\chi(ae_m)$ needs to be zero. Hence there exists for $a \in \mathcal{A}$ some $n \in \mathbb{N}$ such that $\chi(ae_n) = \chi(a)$.

Let $a \in \mathcal{A}$ given by $a_k = \frac{1}{k^2}\mathbb{1}$ for all k and let $n \in \mathbb{N}$ be such that $\chi(ae_n) = \chi(a)$. We define the restricted character $\chi_n(b) = \chi(be_n)$ and claim that $\chi = \chi_n$ for all $b \in \mathcal{A}$. This follows from the obervation that χ and χ_n agree on $\ker(\chi)$ as well as on a.

We have therefore obtain an identification $\bigsqcup_{n\in\mathbb{N}} \sigma(\mathcal{A}_n) \cong \sigma(\mathcal{A})$. We need to check that the topologies match. For this, observe first that $\sigma(\mathcal{A}_n)$ forms an open subset of $\sigma(\mathcal{A})$: If $\chi \in \sigma(\mathcal{A}_n)$ then for any $\chi' \in \sigma(\mathcal{A}_m)$ where $m \neq n$ and any $a \in \mathcal{A}_n$ we have

$$|\chi(a) - \chi'(a)| = |\chi(a)|.$$

So if ε is small enough, the neighborhood $\{\chi' : |\chi(a) - \chi'(a)| < \varepsilon\}$ consists only of characters coming from $\mathcal{A}_n$.

Thus, we now only need to show that the topology on $\sigma(\mathcal{A}_n)$ as a subset of $\sigma(\mathcal{A})$ coincides with the already given topology on $\sigma(\mathcal{A}_n)$. As the former is given by a continuity requirement on a larger set of functions than the latter one, one topology is finer than the other. This implies that they agree by the same argument as earlier as both topologies are compact, Hausdorff topologies.

b) Let us check injectivity first. If $\chi_1 \circ L = \chi_2 \circ L$ for $\chi_1, \chi_2 \in \beta\mathbb{N}$ then we evaluate these on $a \in \prod_{n\in\mathbb{N}} \mathcal{A}_n$ given by $a_n = \alpha_n \mathbb{1}$ for all $n \in \mathbb{N}$ where $\alpha \in \ell^\infty(\mathbb{N})$. Then

$$\chi_1 \circ L(a) = \chi_1(\alpha L(\mathbb{1})) = \chi_1(\alpha)$$

and similarly for χ_2 which shows that $\chi_1 = \chi_2$.

Note that for any $a \in \prod_{n\in\mathbb{N}} \mathcal{A}_n$

$$\|L(a)\|_\infty = \sup_{n\in\mathbb{N}} |\chi_n(a_n)| \leq \sup_{n\in\mathbb{N}} \|a_n\| = \|a\|_\infty.$$

This gives the desired continuity.

c) The element $a_n \in \mathcal{A}_n$ is nilpotent and the smallest positive integer k with $a_n^k = 0$ is $k = n$. This implies that any character χ on $\mathcal{A}_n$ needs to vanish on a_n as $\chi(a_n)^n = \chi(a_n^n) = 0$. The same argument applies to any other nilpotent element and in particular to the cosets of $e_2, \ldots, e_{n-1}$. Thus, by linearity the character is uniquely determined by its value at the identity (which is the coset of e_0). The only non-trivial character on $\mathcal{A}_n$ is thus given by the character

$$\mathcal{A}_n \to \mathcal{A}_n/\mathcal{A}_n e_1 = \mathbb{C}.$$

This shows that $\sigma(\mathcal{A}_n)$ is a one-point set.

To show that b has spectral radius 0 we may evaluate b at any character χ and show that $\chi(b) = 0$. Note that by part a) there is a character χ_m on some $\mathcal{A}_m$ such that $\chi_m(b_m) = \chi(b)$. Since b_m is nilpotent, $\chi_m(b_m) = 0$ and so b has spectral radius zero. Note that if b were nilpotent and $b^k = 0$ then $b_n^k = 0$ for every n or equivalently $a_n^k = 0$ for every n. This contradicts the first sentence in c).

The last part of c) follows directly from b).

d) Let us begin by computing the norm of any coset $a \in \mathcal{A}_n$. We claim that $\|a\| = \|\tilde{a}\|_1$ where $\tilde{a}_1$ is the unique representative of a which is a polynomial of degree strictly less than n. Indeed, we certainly have $\|a\| \leq \|\tilde{a}\|_1$ and if $b \in \ell^1(\mathbb{N})$ is any other representative of a then $b - \tilde{a}$ is a power series whose first non-zero coefficient appears at earliest at the index n and so $\|b\|_1 = \|\tilde{a}\|_1 + \|b - \tilde{a}\|_1 \geq \|\tilde{a}\|_1$. This proves the claim.

We use this to compute the norm of $a_n^k \in \mathcal{A}_n$ for $k < n$. Note that a_n^k is the coset of e_k and so $\|a_n^k\| = \|e_k\|_1 = 1$. This also proves that

$$\|a^k\|_\infty = \sup_{n\in\mathbb{N}} \|a_n^k\| = 1.$$

In particular, by the spectral radius formula the spectral radius of a is

$$\lim_{k\to\infty} \sqrt[k]{\|a^k\|_\infty} = 1.$$

We now compute the value of a on a character arising as in b). So let $\chi_1, \chi_2, \ldots$ be a choice and L be constructed as in b). Note that for our particular choice of a

$$L(a) = (\chi_n(a_n))_n = 0$$

and so for any character $\chi' \in \beta\mathbb{N}$ we have $\chi' \circ L(a) = 0$. This proves that the value of any character arising as in b) is zero. However, by Theorem 11.23 the spectral radius of a is given by the maximum over $|\chi(a)|$ for all characters χ on $\prod_{n\in\mathbb{N}} \mathcal{A}_n$. So there must be some character χ with $|\chi(a)| = 1$. This character is not as in b).